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APR 1952

ALGEBRAIC AND LINEAR DEPENDENCE OF AUTOMORPHIC FUNCTIONS IN SEVERAL VARIABLES

By S. BOCHNER

[Received June 13, 1951]

THEOREM 1. *Let $S = S_{2k}$ be a (non-compact) complex co-ordinate space of $2k$ real dimensions, and let S^0 be a domain in it whose closure $\bar{S}^0$ can be covered by a finite number of allowable co-ordinate systems.*

If $k+2$ holomorphic functions

$$f_1, \dots, f_{k+2} \tag{1}$$

on S are such that for every homogeneous polynomial

$$Q_n(w_1, \dots, w_{k+2}) \not\equiv 0 \tag{2}$$

of any degree n the function

$$\phi(P) = Q_n[f_1(P), \dots, f_{k+2}(P)] \tag{3}$$

satisfies the relation

$$\sup_{P \in S} |\phi(P)| \leqslant M^n . \sup_{P \in S^0} |\phi(P)| \tag{4}$$

for a suitable constant $M > 1$, then there exists one such polynomial $Q_n^0(w)$ for which

$$Q_n^0(f_1, \dots, f_{k+2}) \equiv 0. \tag{5}$$

PROOF. It is easily seen that we can select in S a finite number of co-ordinate neighbourhoods

$$U_1, \dots, U_m, \tag{6}$$

each uniformized by the unit sphere

$$r \equiv (z_1 \bar{z}_1 + \dots + z_k \bar{z}_k)^{1/2} < 1, \tag{7}$$

such that the m closed sub-spheres

$$r \leqslant \tfrac{1}{2} \tag{8}$$

together constitute a covering of the closed set $\bar{S}^0$. On the other hand, if $\phi(z) \equiv \phi(z_1, \ldots, z_k)$ is holomorphic in (7), if its power series at the origin

$$\phi(z) = \sum_{n_1+\ldots+n_k \geqslant p} a_{n_1\ldots n_k} z_1^{n_1} \ldots z_k^{n_k} \tag{9}$$

has no terms of order $< p$, and if $|\phi(z)| < 1$ in (7), then in the sub-sphere (8) we have

$$|\phi(P)| < \frac{1}{2^p}\,. \tag{10}$$

This is a trivial extension of Schwarz's lemma from one to several variables [2].

Now, with the given functions (1) we form for fixed n the monomials

$$\phi_A(z) = f_1^{n_1} \ldots f_{k+2}^{n_{k+2}},\ A = 1, \ldots, N,$$

where $n_1 + \ldots + n_{k+2} = n$ and $N = \binom{n+k+1}{k+1}$; and the assertion is that for n sufficiently large, these N monomials are linearly dependent. If this were not so, then for any $p = p_n$ satisfying

$$m\binom{p+k-1}{k} < \binom{n+k+1}{k+1} \tag{11}$$

we could find a linear combination

$$\phi = \sum_{A=1}^{N} C_A\, \phi_A\,(P)$$

such that

$$\sup_{P \in S} |\phi(P)| = 1, \tag{12}$$

and that at the centres of the m neighbourhoods (6) we would have expansions of the form (9). Now, by Schwarz's lemma, we would have inequality (10) everywhere in $\bar{S}^0$, and on the other hand, by (4) and (12), in at least one point of $\bar{S}^0$ we would have

$$|\phi(P)| \geqslant \frac{1}{M^n}\,; \tag{13}$$

and if we compare (10) and (13) we would obtain

$$p \leqslant \frac{\log M}{\log 2} \cdot n. \tag{14}$$

However, for k fixed, and n and p large, we have

$$\binom{n+k+1}{k+1} \sim \frac{1}{(k+1)!}\, n^{k+1}, \quad \binom{p+k-1}{k} \sim \frac{1}{k!}\, p^k,$$

and therefore it is compatible with (1) to have

$$p_n \geqslant C_0\, n^{1+1/k},$$

where C_0 is fixed and n is large. But this is not compatible with (14), and the proof of the theorem is completed.

REMARK. It is obviously possible to give an estimate for a degree n satisfying (5) in terms of the constants $\{k, M, m\}$ only.

Now, let $V = V_{2k}$ be a compact complex co-ordinate space, let $\tilde{V}$ be its (non-compact) universal covering space, and let $\Gamma = \tilde{V}/V$ be the fundamental group. Since V is compact, there is no holomorphic function on it, other than a constant, that is, but on a non-compact $\tilde{V}$ there might be some; and we call a function on $\tilde{V}$ automorphic if for every α in Γ we have

$$f(\alpha P) = \eta_\alpha(P) . f(P), \tag{15}$$

for some suitable factor $\eta_\alpha(P)$, itself also holomorphic on $\tilde{V}$. This is a broader notion of "automorphy" than sometimes conceived of; and an assemblage of functions

$$\{\eta_\alpha(P)\}, \quad \alpha \in \Gamma, \tag{16}$$

will qualify for the role of such factors of automorphy if they satisfy the law of composition

$$\eta_\beta(P) \,.\, \eta_\alpha(\beta P) = \eta_{\alpha\beta}(P) \tag{17}$$

for all pairs of elements $\alpha, \beta \in \Gamma$, and this is all they need satisfy.

THEOREM 2. *If we are given a compact complex space V, its covering space $\tilde{V}$, the fundamental group Γ, and a fixed system of factors* (16) *which satisfy* (17), *then any $k+2$ holomorphic functions on $\tilde{V}$ which are automorphic with regard to* (16) *satisfy an equation* (5).

PROOF. We can choose in $\tilde{V}$ a domain S^0 having the following properties: the union R of S^0 with part of its boundary constitutes a "fundamental domain" in $\tilde{V}$ relative to the group of automorphisms Γ; and there is a finite set of elements $\alpha_1, \ldots, \alpha_p$

of Γ such that the interior of the union of R with its "translations" $\alpha_1(R), \ldots, \alpha_p(R)$ constitutes a neighbourhood S of the closure $\overline{S^0}$. Also $\overline{S^0}$ is compact in S and can be covered by a finite number of holomorphic co-ordinate systems there. This being so, if we denote by M the supremum of $|\eta_\alpha(P)|$ for P in R and $\alpha = \alpha_1, \ldots, \alpha_p$, then all assumptions of Theorem 1 are fulfilled and the conclusion of Theorem 2 follows.

If $\tilde{V}$ is a domain in Euclidean space E_{2k}, and thus is covered by one co-ordinate system, then the elements of Γ can be represented "globally" by functions

$$z_j^\alpha = f_j^\alpha(z_1, \ldots, z_k), \quad j = 1, \ldots, k, \tag{18}$$

and in this case any power of the Jacobian, that is to say the quantity

$$\left[\frac{\partial(z_1^\alpha, \ldots, z_k^\alpha)}{\partial(z_1, \ldots, z_k)}\right]^W, \tag{19}$$

for any constant W, is an admissible factor $\eta_\alpha(z)$, and in ordinary parlance the concept of an automorphic function is restricted to such a situation only. However, such a restriction is unsatisfactory even within the frame-work of nineteenth century analysis. In fact, if $\tilde{V}$ is the entire E_{2k} and Γ consists of ordinary translations, so that V is a multi-torus, then (19) has the constant value 1 and is useless as a "factor", and other factors must be introduced. Such a factor was indeed introduced by Jacobi, and it is the oldest one on record, apparently. It has the form $e^{2\pi i P(z,\alpha)}$, where $P(z, \alpha)$ is linear in the space variables z, and quadratic in the group parameters describing "α", and the automorphic functions (in our general sense) pertaining to such are the Jacobi functions so-called ("fonctions intermédiares"). Jacobi's selection of his factor has been fully vindicated by P. Appell [1], who showed that in case of "multiperiodicity" the most general factor (16) with only property (17) to start with is, in substance, reducible to one of Jacobi's description; and this seems to be the only case known in which

the factor (16) in its generality has been "reduced" to a particular case of itself, by which to analyze and control it.

THEOREM 3. *If S, S^0 are as in Theorem 1; if F is a module of holomorphic functions on S, each bounded; and if for all elements $f(P)$ of F we have*

$$\sup_{P \in S} |f(P)| \leqslant M . \sup_{P \in S^0} |f(P)|, \tag{20}$$

for some fixed M; then for some suitable integer N (which, incidentally, depends only on S, S^0, M), any $N+1$ elements of F are linearly dependent. That is to say, the module has a finite basis of at most N elements.

PROOF. As in the proof of Theorem 1, if the theorem were false we would have an element $f(P)$ whose expansion at the centres of (6) would begin with terms of order $\geqslant p$, where p is any fixed number for which

$$m \binom{p+k-1}{k} < N.$$

If we put $\sup_{P \in S} |f(P)| = 1$, then at a suitable point of $\overline{S^0}$ we would have

$$\frac{1}{M} \leqslant |f(P)| \leqslant \frac{1}{2^p},$$

which however for large p, and hence for large N, is contradictory.

This now leads to the following conclusion.

THEOREM 4. *If V, $\tilde{V}$, $\eta_a(P)$ are as in Theorem 2, then the module of automorphic functions has a finite basis.*

In a previous paper [2] we have established theorems on the existence of finite bases in certain modules of scalars, tensors, tensor densities, and such-like objects on compact spaces, but we note that the conclusion of our present Theorem 4 is contained in the latter paper only for the particular case in which $\tilde{V}$ is a domain and the factor is of the special form (19). Thus Theorem 4, and Theorem 3, are new relative to the statements previously made. We note however that the method of the previous paper will also immediately establish the following

theorem, which in its turn is vastly more general than our present Theorem 3 itself.

THEOREM 5. *If S_n is a (real) separable co-ordinate space, and S^0 a domain in it with compact closure $\overline{S^0}$, and if F is a module of continuous functions on S_n (over real or complex numbers), then in order that F have a finite basis, it is sufficient to know that the following two assumptions are satisfied :* (i) *for a suitable constant $M > 1$ we have*

$$\sup_{P \in S} |f(P)| \leqslant M. \sup_{P \in S^0} |f(P)|$$

for all $f \in F$, and (ii) *with each point P of S there is associable a neighbourhood U, and a sub-neighbourhood $U' \subset U$, such that any set of elements in F which are equally bounded in U are equally uniformly continuous in U'.*

As in the previous paper, we can also generalize Theorem 5 from scalars f to arbitrary tensor densities of any description, and we are not stating this generalization because we have no fruitful interpretation for it, for the present. But we would like to make the observation that the two theorems on automorphic functions, that sufficiently many of them are linearly dependent and that $k+2$ of them are polynomially dependent, — are very different propositions, first impressions notwithstanding, and that further progress in the theory may be expected to come more from a study of the linear dependences between objects other than scalars, rather than from a further study of the algebraic dependences between the scalars themselves.

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A NOTE ON LATTICE COVERINGS

24 APR 1952

By R. P. BAMBAH and K. F. ROTH

[Received August 17, 1951]

1. Let $\mathcal{K}$ be a symmetrical convex body with centre at the origin O in the n-dimensional space R_n. We denote by $\mathcal{K}+X$, the body with centre at the point X which can be obtained from $\mathcal{K}$ by a translation. A lattice Λ is said to be a *covering lattice* for $\mathcal{K}$ if every point in R_n belongs to a body of the type $\mathcal{K}+X$, where X is a point of Λ. The ratio of the total volume of the bodies to the whole space is obviously $V(\mathcal{K})/d(\Lambda)$, where $V(\mathcal{K})$ is the volume of $\mathcal{K}$ and $d(\Lambda)$ is the determinant of Λ. *The density* $\theta(\mathcal{K})$ *of the most economical lattice covering* of space by $\mathcal{K}$ is defined to be the lower bound of $V(\mathcal{K})/d(\Lambda)$ for all covering lattices Λ of $\mathcal{K}$. The *covering constant* $c(\mathcal{K})$ of $\mathcal{K}$ is defined to be the upper bound of the determinant $d(\Lambda)$ of the covering lattices for $\mathcal{K}$. Plainly

$$\theta(\mathcal{K}) = V(\mathcal{K})/c(\mathcal{K}). \tag{1}$$

Also, it is clear from the definition of $\theta(\mathcal{K})$ that

$$\theta(\mathcal{K}) \geqslant 1. \tag{2}$$

On the other hand it was shown by Hlawka [3, p.87] that for all convex bodies with a centre

$$\theta(\mathcal{K}) \leqslant n^n. \tag{3}$$

Rogers [4] improved (3) to

$$\theta(\mathcal{K}) \leqslant 3^{n-1}. \tag{4}$$

The object of this note is to prove

THEOREM A. *Let $\mathcal{K}$ be any convex body in R_n, which is symmetrical about the coordinate planes. Then*

$$\theta(\mathcal{K}) \leqslant \frac{\pi}{3\sqrt{3}} \prod_{\nu=2}^{n} \left(1-\frac{1}{\nu}\right)^{1-\nu} = \frac{\pi}{3\sqrt{3}} \frac{n^n}{n!}. \tag{5}$$

The term on the right hand side in (5) is asymptotically equal to $e^n (\pi/54n)^{1/2}$ as $n \to \infty$.

It is clear from the definition of the covering constant that if $\mathcal{K}'$ is a body contained in $\mathcal{K}$, then $c(\mathcal{K}') \leqslant c(\mathcal{K})$. We call an n-dimensional body $\mathcal{P}$ an n-dimensional space-filling body if $c(\mathcal{P}) = V(\mathcal{P})$, where $V(\mathcal{P})$ is the volume of $\mathcal{P}$. Then it is clear that if $\mathcal{K}$ contains an n-dimensional space-filling body $\mathcal{P}$, then

$$\theta(\mathcal{K}) = V(\mathcal{K})/c(\mathcal{K}) \leqslant V(\mathcal{K})/c(\mathcal{P}) = V(\mathcal{K})/V(\mathcal{P}), \tag{6}$$

so that it will suffice for the proof of Theorem A to prove

THEOREM B. *If $\mathcal{K}$ satisfies the conditions of Theorem* A. *then $\mathcal{K}$ contains an n-dimensional space-filling body $\mathcal{P}$ such that*

$$V(\mathcal{P}) \geqslant \frac{3\sqrt{3}}{\pi} V(\mathcal{K}) \prod_{\nu=2}^{n} \left(1 - \frac{1}{\nu}\right)^{\nu-1}. \tag{7}$$

Our thanks are due to Dr. C. A. Rogers whose suggestions enabled us to simplify the proof of Lemma 2. We are also grateful to Prof. H. Davenport for useful comments on the manuscript.

2. LEMMA 1. *Suppose $\mathcal{K}_1$ is an $(n-1)$ dimensional space-filling body defined by the relation*

$$G(x_1, x_2, \ldots, x_{n-1}) \leqslant 1.$$

Then, for any $t > 0$, the body $\mathcal{K}_2$ defined by

$$\max \left\{ G(x_1, \ldots, x_{n-1}), \frac{|x_n|}{t} \right\} \leqslant 1$$

is an n-dimensional space-filling body.

PROOF. Obvious.

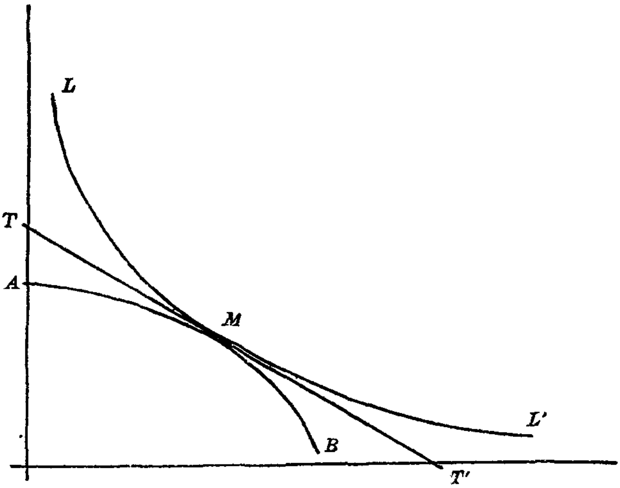
LEMMA 2. *Let $y = f(x)$ be a non-negative function of x which decreases in the closed interval $(0, a)$, $a > 0$, and which satisfies the inequality*

$$f\{(1-\theta)x_1 + \theta x_2\} \geqslant (1-\theta) f(x_1) + \theta f(x_2), \tag{8}$$

for $0 \leqslant \theta \leqslant 1$ and x_1 and x_2 in $(0, a)$. Then

$$\max_{0 \leqslant x \leqslant a} xy^{n-1} \geqslant \left(1 - \frac{1}{n}\right)^{n-1} \int_0^a y^{n-1}\, dx. \tag{9}$$

PROOF.



Let AB be the curve $y = f(x)$ between $x = 0$ and $x = a$. From the conditions on $f(x)$ it is easy to see that the curve AB is continuous and convex upwards. Let M be a point on AB where xy^{n-1} assumes its greatest value, α say. Let LML' be the curve $xy^{n-1} = \alpha$. This curve is concave upwards and at every point has a unique tangent whose slope is negative. Let TMT' be the tangent to LML' at M. This tangent meets the axes at points T and T' in the first quadrant as shown in the figure. Also every point other than M of TMT' lies strictly below LML'.

We now assert that TMT' is a tac-line to the curve AB, i.e. AB lies below TMT'. If not, there is a point N of AB which lies strictly above TMT'. Then because of the convexity upwards of AB every point on the line segment MN lies below AB. Consequently for coordinates (x, y) of every point on MN we have

$$xy^{n-1} \leqslant \alpha. \tag{10}$$

On the other hand, since TMT' is the tangent at M to LML' and N lies strictly above TMT', all points on the segment

MN sufficiently close to M lie strictly above the curve LML'. So that for the coordinates of points on (x, y) sufficiently near to M, we have

$$xy^{n-1} > a. \tag{11}$$

This is a contradiction to (10) and our assertion about TMT' is true, i.e. the curve AB lies below TMT', and we have

$$\int_{AB} y^{n-1}\, dx \leqslant \int_{TMT'} y^{n-1}\, dx. \tag{12}$$

Also, since every point on TMT' lies below LML', we have

$$\max_{(x,y)\text{ on } TMT'} xy^{n-1} = a. \tag{13}$$

From (12) and (13) we have

$$\frac{\max\limits_{(x,y)\text{ on } AB} xy^{n-1}}{\int_{AB} y^{n-1}\, dx} \geqslant \frac{\max\limits_{(x,y)\text{ on } TMT'} xy^{n-1}}{\int_{TMT'} y^{n-1}\, dx}. \tag{14}$$

The equation of TT' is of the form

$$y + mx = mc,$$

where m and c are positive numbers. Therefore

$$\max_{(x,y)\text{ on } TMT'} xy^{n-1} = \max_{0 \leqslant x \leqslant c} m^{n-1}x(c-x)^{n-1}$$

$$= \frac{m^{n-1}}{n} c^n \left(1 - \frac{1}{n}\right)^{n-1}, \tag{15}$$

and

$$\int_{TT'} y^{n-1}\, dx = \int_0^c m^{n-1}(c-x)^{n-1}\, dx = \frac{m^{n-1}}{n} c^n. \tag{16}$$

The lemma follows from (14), (15) and (16).

Lemma 3. (Brunn-Minkowski [1]). *Let $\mathcal{K}_x$ be the $(n-1)$-dimensional section of $\mathcal{K}$ by the hyper-plane $x_n = x$. Let V_x be the $(n-1)$-dimensional volume of $\mathcal{K}_x$. Write*

$$f(x) = (V_x)^{1/(n-1)}. \tag{17}$$

Then, if $0 \leqslant \theta \leqslant 1$ and the hyper-planes $x_n = x_1$ and $x_n = x_2$ meet $\mathcal{K}$,

$$f\{(1-\theta)x_1 + \theta x_2\} \geqslant (1-\theta) f(x_1) + \theta f(x_2). \tag{18}$$

3. Proof of Theorem B. Let $x_n = \pm a$ be the tac-planes of $\mathcal{K}$, parallel to $x_n = 0$. It is clear from the symmetry about $x_n = 0$ of $\mathcal{K}$ that the function $f(x)$, as defined in Lemma 3, is an even function of x. It follows from (18) that $f(x)$ satisfies all the conditions of Lemma 2. Consequently there exists a real number t with $0 \leqslant t \leqslant a$ such that

$$\begin{aligned} t V_t = t \{f(t)\}^{n-1} &\geqslant \left(1 - \frac{1}{n}\right)^{n-1} \int_0^a \{f(x)\}^{n-1}\, dx \\ &= \left(1 - \frac{1}{n}\right)^{n-1} \int_0^a V_x\, dx \qquad (19) \\ &= \tfrac{1}{2}\left(1 - \frac{1}{n}\right)^{n-1} V(\mathcal{K}). \end{aligned}$$

We shall now prove the theorem by induction on n, the number of dimensions of $\mathcal{K}$. We suppose that the theorem is true in the $(n-1)$-dimensional space. Then since $\mathcal{K}$ is an $(n-1)$-dimensional convex body which is symmetrical about the planes $x_1 = 0$, $x_2 = 0$, ..., $x_{n-1} = 0$, it contains an $(n-1)$-dimensional space-filling body $\mathcal{P}_{n-1}$, whose $(n-1)$-dimensional volume $V(\mathcal{P}_{n-1})$ satisfies the relation

$$V(\mathcal{P}_{n-1}) \geqslant \frac{3\sqrt{3}}{\pi} V_t \prod_{\nu=2}^{n-1} \left(1 - \frac{1}{\nu}\right)^{\nu-1}. \qquad (20)$$

By Lemma 1, the n-dimensional body $\mathcal{P}_n$ formed by taking the convex cover of $\mathcal{P}_{n-1}$ and its image in the plane $x_n = 0$ is an n-dimensional space-filling body contained in $\mathcal{K}$. Also, because of (20) and (19), its n-dimensional volume satisfies the relation

$$\begin{aligned} V(\mathcal{P}_n) = 2t\, V(\mathcal{P}_{n-1}) &\geqslant \frac{3\sqrt{3}}{2\pi} 2t V_t \prod_{\nu=2}^{n-1} \left(1 - \frac{1}{\nu}\right)^{\nu-1} \\ &\geqslant \frac{3\sqrt{3}}{2\pi} V(\mathcal{K}) \prod_{\nu=2}^{n} \left(1 - \frac{1}{\nu}\right)^{\nu-1}, \end{aligned}$$

so that in order to complete the proof we have only to show that the theorem is true when $n = 2$.

It is well-known that symmetrical hexagons in two dimensions are space-filling bodies. It also follows from certain results of Sas [5] and Dowker [2] that every convex body $\mathcal{K}$ with a centre in two dimensions contains a symmetrical hexagon $\mathcal{P}_2$ such that

$$V(\mathcal{P}_2) \geqslant \frac{3\sqrt{3}}{2\pi} V(\mathcal{K}),$$

where $V(\mathcal{P}_2)$ and $V(\mathcal{K})$ are the areas of $\mathcal{P}_2$ and $\mathcal{K}$ respectively. This completes the proof of the theorem.

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ON THE EXISTENCE OF A SOLUTION OF AN INFINITE DIFFERENTIAL SYSTEM

By B. VISWANATHAM

[Received January 3, 1951 and, in revised form, April 2, 1951]

1. The aim of this note is to prove the existence of a solution of an infinite differential system,

$$\text{(A)} \qquad \frac{d}{dx}\{y^i(x)\} = f^i(x, y^1, y^2, \ldots)\,; \qquad (i = 1, 2, 3, \ldots)$$

with the initial conditions, $y^i(\xi) = \eta^i$, $(i = 1, 2, 3, \ldots)$ subject to the following hypotheses:

(1) *The functions f^i are bounded for each i in the region of the infinite dimensional space defined by*

$$a \leqslant x \leqslant b\,; \quad -\infty < y^i < +\infty\,; \quad (i = 1, 2, 3, \ldots),$$

i.e. $|f^i(x, y^1, \ldots)| \leqslant M^i$ *in the above region where M^i is a certain constant for each i;*

(2) *The functions f^i are continuous at all points of the above region in the sense of the definition of continuity of functions of an infinite number of variables to be given below.*

We shall call this the main theorem. From the above theorem we shall deduce an analogue of Perron's generalized existence theorem for the case of a single equation, and Müller's extension of this theorem to the case of a finite system, in §4. In §5, we shall deduce a theorem for a finite domain as a special case of our main theorem. Further in §6, a case is discussed in which the boundedness of the functions f^i is relaxed.

Aurel Wintner [1] in 1925 proved the existence of a solution by the method of sections in the case of the right hand members of the system being regular power series involving an infinite

number of variables. Later in 1933, D. C. Lewis [2] proved the existence in the case of more general functions by the method of successive approximations. Lewis' paper contains a bibliography which the author thinks is complete. The present note aims at proving the existence by the properties of families of uniformly bounded equi-continuous functions. The method employed is essentially that found in Mcshane's book [3] with the necessary extensions applicable to the case of infinite systems. The crux of the method consists in applying Cantor's principle of diagonal selection to obtain a sequence of natural numbers such that, when a ranges over this set, each member of an infinite system of sequences uniformly converges on the given interval.

The following two well-known lemmas are used in the course of the proof.

LEMMA 1. *From an infinite family of uniformly bounded equi-continuous functions of a single variable, defined in a closed, finite interval it is possible to extract a sequence uniformly converging on the interval.*

LEMMA 2. *If* $\{\chi_n(t)\}$ *is a sequence of uniformly bounded, integrable functions converging to* $\chi(t)$, *then*

$$\lim_{n\to\infty}\int \chi_n(t)\,dt = \int \lim_{n\to\infty} \chi_n(t)\,dt = \int \chi(t)\,dt.$$

2. DEFINITION OF CONTINUITY OF A FUNCTION OF AN INFINITE NUMBER OF INDEPENDENT VARIABLES. Given a sequence of points

$$X_n = (x_n^1, x_n^2, x_n^3, \ldots), \qquad n = 1, 2, 3, \ldots,$$

in the domain in which a function ϕ is defined, the function ϕ is said to be continuous at the point $X_0 = (x_0^1, x_0^2, x_0^3, \ldots)$ if for every sequence in which $\lim_{n\to\infty} x_n^p = x_0^p$, $(p = 1, 2, 3, \ldots)$, we always have

$$\lim_{n\to\infty} \phi(x_n^1, x_n^2, x_n^3, \ldots) = \phi(x_0^1, x_0^2, x_0^3, \ldots).$$

In what follows we shall suppose that all the functions f^i are continuous in the above sense. Here the definition of continuity is so framed that if $x^1, x^2, x^3, \dots x^n$, are continuous functions of a parameter t then $\phi(t) = \phi(x^1, x^2, \dots)$ is a continuous function of t.

3. Proof of the theorem. The differential system (A) with its initial conditions can be transformed into an infinite system of integral equations,

$$\text{(B)} \qquad y^i(x) = \eta^i + \int_{\xi}^{x} f^i\{t, y^1(t), y^2(t), \dots\}\, dt, \quad (i = 1, 2, 3, \dots).$$

In order that the transformed integral equations have any meaning, we have to show that the functions f^i are integrable. If all the $y^i(x)$, $(i = 1, 2, \dots)$, are continuous on the given interval, in view of the definition of continuity given above and condition (2) of §1, the functions f^i are continuous functions of x, which can be easily verified, and hence they are integrable. We shall subsequently show that all $y^i(x)$ are continuous and differentiable on the given interval.

Define a function $U_n(x)$ as follows, for every n which is a positive integer,

$$U_n(x) = \begin{cases} x + \dfrac{1}{n}, & \text{if } a \leqslant x < \xi - \dfrac{1}{n}; \\ \xi, & \text{if } \xi - \dfrac{1}{n} \leqslant x < \xi + \dfrac{1}{n}; \\ x - \dfrac{1}{n}, & \text{if } \xi + \dfrac{1}{n} \leqslant x \leqslant b. \end{cases}$$

Thus defined, $U_n(x)$ has the following important properties.

(a). $U_n(x)$ is continuous throughout.

(b). For any two points x_1 and x_2 of the given interval

$$|U_n(x_1) - U_n(x_2)| \leqslant |x_1 - x_2|.$$

(c). $$|U_n(x) - x| \leqslant \frac{1}{n}.$$

The above properties can be easily verified.

For a chosen value of n consider a sequence of functions $g_n^i(x)$ defined by the relation,

$$\text{(C)}\quad g_n^i(x) = \eta^i + \int_\xi^{U_n(x)} f^i\{t, g_n^1(t), g_n^2(t), \ldots\}\, dt, \qquad i = 1, 2, 3, \ldots.$$

First, we have to show that the functions can be defined thus. Now if x lies in the range $(\xi - 1/n,\ \xi + 1/n)$ the value of $U_n(x)$, according to the definition, is ξ and the integral vanishes. For this range, $g_n^i(x)$ are simply the constants η^i. It can be shown by induction that they can be defined throughout the whole interval. If the functions $g_n^i(x)$ are defined by the above relation (C) in the range $[\xi, \xi + (j-1)/n]$, where j is a positive integer, they are necessarily continuous on this interval, being integrals. Hence all

$$f^i\{x, g_n^1(x), g_n^2(x), \ldots\}, \qquad i = 1, 2, 3, \ldots,$$

are integrable over the sub-interval. Now, if x lies in the interval $[\xi + (j-1)/n, \xi + j/n]$, the value of $U_n(x)$ there is $[\xi + (j-1)/n + \delta - 1/n]$, where δ is a positive real number less than $1/n$ according to the definition of $U_n(x)$. The range of integration is then from ξ to $\xi + (j-2)/n + \delta$, and this range is obviously contained in the range $[\xi, \xi + (j-1)/n]$ as $\delta \leqslant 1/n$. Hence, if the value of the integral is defined for the range $[\xi, \xi + (j-1)/n]$ it is also defined for the range $(\xi, \xi + j/n)$, and so the functions $g_n^i(x)$ are defined in this range. By regular steps we can extend the range of definition to any length to the right. Similarly $g_n^i(x)$ can be defined in any range to the left of ξ, and throughout the whole range the functions $f^i(x, g_n^1, g_n^2, \ldots)$ are integrable, in fact continuous in x for any chosen value of n.

Defined thus, $g_n^i(x)$ form a uniformly bounded sequence for each i, because

$$|g_n^i(x)| \leqslant |\eta^i| + \left| \int_\xi^{U_n(x)} f^i(t, g_n^1, g_n^2, \ldots)\, dt \right|$$
$$\leqslant |\eta^i| + \left| \int_a^{b} M^i\, dt \right|$$
$$\leqslant |\eta^i| + |b - a|\, M^i.$$

Further, they are equi-continuous relative to n, because

$$|g_n^i(x_1) - g_n^i(x_2)| = \left| \int_{U_n(x_1)}^{U_n(x_2)} f^i(t, g_n^1, g_n^2, \ldots)\, dt \right|$$

$$\leqslant |U_n(x_1) - U_n(x_2)|\, M^i$$

$$\leqslant |x_1 - x_2|\, M^i, \qquad \text{on account of the property } (b) \text{ of } U_n(x);$$

$$\leqslant \epsilon \text{ if } |x_1 - x_2| \leqslant \frac{\epsilon}{M^i} = \delta^i,$$

(where M^i is the upper bound of the function f^i).

If η^i and M^i are bounded the g_n^i are bounded and equi-continuous with respect to i also.

From this doubly infinite family of equi-continuous functions we can extract sequences $\{g_a^i\}$ such that for each fixed i, $g_a^i(x)$ uniformly converges to a continuous function $y^i(x)$ as $a \to \infty$ in the following way.

Take first the uniformly bounded sequence of equi-continuous functions

$$g_1^1,\ g_2^1,\ g_3^1,\ g_4^1,\ g_5^1, \ldots .$$

From this it is possible to extract a sub-sequence unifomly converging on the interval. Let the sequence be

$$g_{1,1}^1,\ g_{1,2}^1,\ g_{1,3}^1,\ g_{1,4}^1,\ g_{1,5}^1, \ldots \tag{1}$$

uniformly converging to y^1.

Now change the superscript from 1 to 2 and consider the sequence

$$g_{1,1}^2,\ g_{1,2}^2,\ g_{1,3}^2,\ g_{1,4}^2,\ g_{1,5}^2, \ldots .$$

This need not be uniformly convergent. But from this it is possible to extract a sub-sequence,

$$g_{2,1}^2,\ g_{2,2}^2,\ g_{2,3}^2,\ g_{2,4}^2,\ g_{2,5}^2, \ldots, \tag{2}$$

uniformly converging to y^2. The subscripts of (2) are drawn from the subscripts of (1) and hence for these subscripts, both

the sequences (1) and (2) are uniformly convergent to y^1 and y^2 respectively.

Change again the superscript from 2 to 3 and consider the sequence

$$g^3_{2,1},\ g^3_{2,2},\ g^3_{2,3},\ g^3_{2,4},\ g^3_{2,5},\ \ldots .$$

This again need not be uniformly convergent. But from this it is possible to extract a sub-sequence,

$$g^3_{3,1},\ g^3_{3,2},\ g^3_{3,3},\ g^3_{3,4},\ g^3_{3,5},\ \ldots \qquad (3)$$

uniformly converging to y^3. The subscripts of (3) are contained in the subscripts of (2) and (1) and hence ensure the uniform convergence of (1), (2) and (3) simultaneously.

If the differential system were a finite one, this process would stop when we have selected a sequence of subscripts which ensures the uniform convergence of every one of the finite number of the sequences. But if it is an infinite system, we apply Cantor's principle of diagonal selection to get the set of subscripts which ensures the uniform convergence of every one of the infinitely many sequences.

Out of the above sequences of subscripts of (1), (2), (3), ... select the following:

$$(1, 1);\ (2, 2);\ (3, 3);\ (4, 4);\ (5, 5);\ \ldots .$$

For this sequence of subscripts every one of the sequences is uniformly convergent, for, take the mth sequence and out of it consider the sub-sequence consisting of the following terms

$$g^m_{1,1},\ g^m_{2,2},\ g^m_{3,3},\ g^m_{4,4},\ g^m_{5,5},\ \ldots .$$

From the method of selection, it is clear that except for the first $(m-1)$ terms, this sequence is contained in the mth uniformly convergent sequence. So, this too converges to the same limit y^m uniformly for each fixed m.

We shall show that these limits satisfy the differential system.

Let a be an element which ranges over the set of the finally chosen subscripts. As $a \to \infty$ each one of the sequences $\{g_a^i(x)\}$ tend to their respective limits $y^i(x)$, $(i = 1, 2, 3, \dots)$. Now consider

$$g_a^i(x) = \eta^i + \int_\xi^{U_a(x)} f^i \{t, g_a^1(t), g_a^2(t), g_a^3(t), \dots\} \, dt$$
$$= \eta^i + \int_\xi^x f^i (t, g_a^1, g_a^2, \dots) \, dt + \int_x^{U_a(x)} f^i(t, g_a^1, g_a^2, \dots) dt.$$

Proceeding to the limit, the last member of the right hand side tends to zero as f^i are bounded and $|U_a(x) - x| \leqslant 1/a \to 0$. Hence

$$\lim_{a \to \infty} g_a^i(x) = \eta^i + \lim_{a \to \infty} \int_\xi^x f^i(t, g_a^1, g_a^2, g_a^3, \dots) \, dt.$$

Now $\phi_a^i(t) = f^i\{t, g_a^1(t), g_a^2(t), g_a^3(t), \dots\}$ are uniformly bounded by hypotheses.

Applying Lemma 2 and the definition of continuity, we at once get, as $\lim\limits_{a \to \infty} \phi_a^i(t) = \phi^i(t) = f^i\{t, y^1(t), y^2(t), \dots\}$ exists, where $\phi(t)$ is continuous in t,

$$y^i(x) = \eta^i + \int_\xi^x f^i\{t, y^1(t), y^2(t), \dots\} \, dt, \quad (i = 1, 2, 3, \dots).$$

All the y^i are continuous and differentiable as they are all expressed as integrals of continuous functions. Differentiating both sides with respect to x, we get

$$\frac{d}{dx} \{y^i(x)\} = f^i(x, y^1, y^2, y^3, \dots), \qquad (i = 1, 2, 3, \dots).$$

Thus the theorem is proved.

4. Theorem. *In the above differential system suppose that the following conditions are satisfied.*

There exist functions $L^i(x)$ and $U^i(x)$ continuous on the interval $\xi \leqslant x \leqslant b$ and satisfying the following relations:

(a). *All functions* $f^i(x, \zeta^1, \zeta^2, \zeta^3, \dots)$ *are continuous in the region defined by* $\xi \leqslant x \leqslant b$ *and* $l^i(x) \leqslant \zeta^i \leqslant U^i(x)$, $(i = 1, 2, 3, \dots)$.

(b). $l^{i\prime}(x)$ *and* $U^{i\prime}(x)$ *exist and satisfy the relations* (i) *and* (ii) *for* $l^i(x) \leqslant \zeta^i \leqslant U^i(x)$, $(i = 1, 2, 3, \dots)$:

(i) $l^{i\prime}(x) \leqslant \min_\zeta f^i(x, \zeta^1, \zeta^2, \dots, \zeta^{i-1}, l, \zeta^{i+1}, \dots)$,

(ii) $U^{i\prime}(x) \geqslant \max_\zeta f^i(x, \zeta^1, \zeta^2, \dots, \zeta^{i-1}, U, \zeta^{i+1}, \dots)$.

(c). $l^i(\xi) = U^i(\xi) = \eta^i$, $(i = 1, 2, 3, \dots)$.

Then there is at least one solution $y^i(x)$ *of the differential system such that* $y^i(\xi) = \eta^i$ *and* $l^i(x) \leqslant y^i \leqslant U^i(x)$; $(i = 1, 2, 3, \dots)$.

PROOF. We define a new function $\phi^i(x, y^1, y^2, \dots)$ as follows: $\phi^i(x, y^1, y^2, y^3, \dots) = f^i(x, \bar{y}^1, \bar{y}^2, \bar{y}^3, \dots)$, where

$$\bar{y}^i = \begin{cases} y^i \text{ if } l^i(x) \leqslant y^i \leqslant U^i(x); \\ U^i \text{ if } y^i > U^i; \\ l^i \text{ if } y^i < l^i. \end{cases}$$

Then the functions ϕ^i satisfy the requirements of the main theorem, and there is at least one solution $y^i(x)$ satisfying the differential system, such that $y^i(\xi) = \eta^i$,

$$\frac{dy^i}{dx} = \phi^i(x, y^1, y^2, \dots), \qquad (i = 1, 2, 3, \dots).$$

We have to show that for each value of i, $l^i(x) \leqslant y^i \leqslant U^i(x)$.

Suppose the inequality is not satisfied for a certain value k of i in the greatest open interval $\xi \leqslant c < x < d$. Then $y^k < l^k$ in this interval. But by definition of $\bar{y}^k$ we have $\bar{y}^k = l^k$ if $y^k < l^k$ and the value of ϕ^k becomes $f^k(x, y^1, y^2, y^3, \dots, y^{k-1}, l^k, y^{k+1}, \dots) = y^{k\prime}$. By (i) of (b), $l^{k\prime}(x) \leqslant y^{k\prime}(x)$. Therefore the function $l^k(x) - y^k(x)$ is non-increasing in this interval, and since its value at $x = c$ is

zero on account of its continuity, it is negative throughout; or $l^k(x) \leqslant y^k(x)$ in the interval, contradicting our assumption that $y^k < l^k$.

Similarly it can be shown that $U^k(x) \geqslant y^k(x)$ throughout, for $k = 1, 2, 3, \ldots$, and the theorem is proved.

5. We have the following particular case of the main theorem.

If f^i are continuous in the infinite dimensional space defined by $|x-\xi| \leqslant a$ and $|y^i-\eta^i| \leqslant C$ then, there is at least one integral curve of the differential system passing through the point $(\xi, \eta^1, \eta^2, \eta^3, \ldots)$ and continuous in the region $|y^i - \eta^i| \leqslant C$ and $|x - \xi| \leqslant \alpha$, where $\alpha = \min(a, C/M)$ and M is the uniform upper bound of the functions f^i in the region defined.

Proof. Define f^i as $f^i(x, \bar{y}^1, \bar{y}^2, \ldots)$, where

$$\bar{y}^i = \begin{cases} y^i \text{ if } \eta^i - C \leqslant y^i \leqslant y^i + C; \\ \eta^i + C \text{ if } y^i > \eta^i + C; \\ \eta^i - C \text{ if } y^i < \eta^i - C. \end{cases}$$

Then the functions f^i satisfy the conditions of the general theorem. Hence there is at least one integral curve passing through the point $(\xi, \eta^1, \eta^2, \eta^3, \ldots)$. Further

$$\begin{aligned} |y^i(x) - \eta^i| &= \left| \int_\xi^x f^i(t, y^1, y^2, \ldots)\, dt \right| \\ &\leqslant |x - \xi|\, M \leqslant \alpha\, M \\ &\leqslant \frac{C}{M} \times M = C. \end{aligned}$$

6. We shall now consider a simple case where f^i are apparently not bounded, but still we get solutions

Consider the differential system,

$$\frac{dy^i}{dx} + \lambda^i y^i = f^i(x, y^1, y^2, \ldots),$$

with the initial conditions $y^i(0) = 0$ *and subject to the restrictions of the main theorem.*

Further λ^i *are positive and increasing numbers not necessarily bounded.*

An equivalent formulation is

$$y^i(x) = \int_0^x \exp[-\lambda^i(x-t)]\, f^i\,(t, y^1, y^2, y^3, \ldots)\, dt.$$

In order to prove the existence of a solution, we can proceed as in Theorem in §1, and it is enough to prove that for each fixed value of i, the sequence of functions

$$g_n^i(x) = \int_0^{U_n(x)} \exp\,[-\lambda^i\{U_n(x)-t\}]\, \phi_n^i(t)\, dt$$

is uniformly bounded and equi-continuous, where

$$f^i\{t, g_n^1(t), g_n^2(t), \ldots\}$$

is written as $\phi_n^i(t)$ for brevity.

That they are uniformly bounded follows from the fact that $\{U_n(x) - t\}$ is positive and ϕ_n^i are bounded.

The equi-continuity can be demonstrated as follows :

$$|g_n^i(x_2) - g_n^i(x_1)| = \left| \int_0^{U_n(x_2)} \exp\,[-\lambda^i\{U_n(x_2) - t\}]\, \phi_n^i(t)\, dt - \right.$$
$$\left. - \int_0^{U_n(x_1)} \exp\,[-\lambda^i\{U_n(x_1) - t\}]\, \phi_n^i(t)\, dt \right|$$
$$= \left| \{\exp[-\lambda^i U_n(x_2)] - \exp[-\lambda^i U_n(x_1)]\} \int_0^{U_n(x_1)} \exp(\lambda^i t)\, \phi_n^i(t)\, dt + \right.$$
$$\left. + \int_{U_n(x_1)}^{U_n(x_2)} \exp\,[-\lambda^i\{U_n(x_2) - t\}]\, \phi_n^i(t)\, dt \right|$$

$$\leqslant M^i \Big[|\exp[-\lambda^i U_n(x_2)] - \exp[-\lambda^i U_n(x_1)]| \Big| \int_0^{U_n(x_1)} |\exp[\lambda^i t]| dt +$$

$$+ |\exp[-\lambda^i U_n(x_2)]| \Big| \int_{U_n(x_1)}^{U_n(x_2)} |\exp[\lambda^i t]| dt \Big],$$

where M^i is the upper bound of ϕ_n^i. Further if we suppose that $x_2 \geqslant x_1$ since $U_n(x_2) \geqslant U_n(x_1)$, the above expression can be written as

$$= \frac{M^i}{\lambda^i} \Big[\{\exp[-\lambda^i U_n(x_1)] - \exp[-\lambda^i U_n(x_2)]\} \{\exp[\lambda^i U_n(x_1)] - 1\} +$$

$$+ \exp[-\lambda^i U_n(x_2)] \{\exp[\lambda^i U_n(x_2)] - \exp[\lambda^i U_n(x_1)]\} \Big]$$

$$= \frac{M^i}{\lambda^i} \Big[2 - 2\exp[-\lambda^i \{U_n(x_2) - U_n(x_1)\}] - \lambda^i \int_{U_n(x_1)}^{U_n(x_2)} \exp(-\lambda^i t)\, dt \Big].$$

Now, since $|U_n(x_2) - U_n(x_1)| \leqslant |x_2 - x_1|$ and $|\exp[-\lambda^i t]| \leqslant 1$, the above expression is

$$\leqslant \frac{M^i}{\lambda^i} \Big[2 - 2\exp[-\lambda^i |x_2 - x_1|] + \lambda^i |x_2 - x_1| \Big].$$

This expression can be made less than ϵ by taking $|x_1 - x_2| < \delta(\epsilon)$. Clearly ϵ does not depend upon n. Hence $g_n^i(x)$ form a sequence of equi-continuous functions with respect to n for fixed values of i, and the theorem follows as in the main theorem.

N.B.—The more general case of the initial value problem where $y^i(\xi) = \eta^i$, $(i = 1, 2, 3, \ldots)$ is not considered, because the procedure and proof are almost the same as in the above case.

This case is of special importance in the theory of boundary value problems of non-linear parabolic equations. See for instance [4].

Finally, I wish to express my sense of thankfulness to Dr. S. Minakshisundaram for his helpful criticism during the preparation of this paper.

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AN ANALOGUE OF LAURENT'S THEOREM FOR A SIMPLY CONNECTED REGION

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1. Our purpose is to establish the following theorem.

THEOREM I. *Given open disks D_1 and D_2 in the complex-plane such that $D = D_1 \cap D_2$ is non-void, and a function f on D with values in a (non-commutative) Banach algebra over the complex field such that f is holomorphic and reciprocable on D, then f is the product of two functions f_1 and f_2 such that f_1 is holomorphic and reciprocable on D_1 and f_2 is holomorphic and reciprocable on D_2.*

By the statement 'f is reciprocable on D' we mean that for all $z \in D$ the element $f(z)$ of the Banach algebra has a reciprocal $\{f(z)\}^{-1}$ with the property

$$f(z).\{f(z)\}^{-1} = 1 = \{f(z)\}^{-1} \cdot f(z),$$

1 being the multiplicative unit of the algebra. An expository account of Banach algebras will be found in Hille [7].

To see the relationship between Theorem I and the well-known theorem of Laurent, we have only to reinterpret (i) multiplication as addition, and reciprocation as subtraction, (ii) D as the open annulus bounded by concentric circles C_1 and C_2 with C_2 inside C_1, (iii) D_1 as the region inside C_1, and D_2 as that outside C_2 and including the point at infinity, (iv) f as a complex-valued function holomorphic on D and (v) f_1, f_2 as complex-valued functions holomorphic on D_1, D_2 respectively. With this reinterpretation, Therem I reduces to Laurent's theorem.

In Theorem I we make no assumption as to the behaviour of f on the closure of D. The strongest known result so far, due

to H. Cartan (see below), assumes however that f is holomorphic and reciprocable on the closure of D. (Cartan [3, p. 9] states his theorem for matrix-valued functions, but his proof is clearly applicable to functions with values in any Banach algebra over the complex-field). But even with this restriction on the *boundary-behaviour of f, the theorem is far from trivial.* The difficulty in proving it arises from the non-commutativity of multiplication in Banach algebras.

Theorems of this type play a fundamental role in the extension of the Second Problem of Cousin to vector-valued functions. They also provide the starting point of the theory of ideals of holomorphic functions in several complex variables due to H. Cartan [4, 5]. Special theorems of this type corresponding to the case when D is an annulus were employed by Hilbert, Plemelj [8] and G. D. Birkhoff [2] in their studies of the singularities of linear differential equations. One of the writers has succeeded in generalizing some of these results for the annulus.

We shall now turn to the proof and then indicate some obvious extensions.

2. Proof of Theorem I. It will be convenient to use the following abbreviations :

'$\mathscr{H}$' for 'holomorphic';

'$\mathscr{H}^*$' for 'holomorphic and reciprocable';

'$\bar{R}$' for 'the closure of the set R'.

Our proof rests on the following theorem due essentially to H. Cartan [3, p. 9].

Cartan's Theorem. *Given D_1, D_2 and D as in Theorem* I *and an $\mathscr{H}^*$ function f on $\bar{D}$ with values in a Banach algebra over the complex field then f is the product of two functions f_1 and f_2 where f_1 is $\mathscr{H}^*$ on $\bar{D}_1$ and f_2 is $\mathscr{H}^*$ on $\bar{D}_2$.*

We shall also require the following lemma which is an obvious multiplicative analogue of Runge's classical theorem on polynomial approximation [6, p. 302].

LEMMA. *If g is an $\mathscr{H}^*$ function on a simply connected closed region $\overline{R}$ and $\epsilon > 0$, then there exists a function E which is $\mathscr{H}^*$ throughout the complex-plane excepting, of course, the point at infinity, and such that for all $z \in \overline{R}$, $|g(z). \{E(z)\}^{-1} - 1| < \epsilon$.*

Let r_1, r_2 be the radii of D_1 and D_2, and let D_1^n be the (open) disk concentric to D_1 and of radius $r_1 - 1/n$. Let D_2^n be defined similarly.

Since f is $\mathscr{H}^*$ on $\overline{D_1^n} \cap \overline{D_2^n}$, therefore, by Cartan's Theorem

$$f = g_n \,.\, h_n^{-1},$$

where g_n is $\mathscr{H}^*$ on $\overline{D_1^n}$ and h_n is $\mathscr{H}^*$ on $\overline{D_2^n}$. (This equation is meant to be an abbreviation of

$$f(z) = g_n(z) \,.\, \{h_n(z)\}^{-1}, \qquad z \in \overline{D_1^n} \cap \overline{D_2^n}.$$

Whether we write $f = g_n \,.\, h_n$ or $f = g_n \,.\, h_n{}^{-1}$ is clearly a matter of notation only, since the functions we are discussing are reciprocable). Since this holds for all n and

$$D_1^n \cap D_2^n \subset \overline{D_1^{n+1}} \cap \overline{D_2^{n+1}},$$

hence on $\overline{D_1^n} \cap \overline{D_2^n}$,

$$g_n \,.\, h_n{}^{-1} = f = g_{n+1} \,.\, h_{n+1}^{-1},$$

and therefore

$$h_n^{-1} \,.\, h_{n+1} = g_n^{-1} \,.\, g_{n+1} \,.$$

Hence letting

$$F_n = \begin{cases} g_n^{-1} \,.\, g_{n+1}, & \text{on } \overline{D_1^n}, \\ h_n^{-1} \,.\, h_{n+1}, & \text{on } \overline{D_2^n}, \end{cases} \tag{1}$$

we see that F_n is $\mathscr{H}^*$ on $\overline{D_1^n} \cup \overline{D_2^n}$.

Now let $(\delta_n)_1^\infty$ be any sequence such that $\delta_n > 0$ and $\Sigma_1^\infty \delta_n < \infty$. We shall construct a corresponding sequence of

functions E_n all of which are $\mathscr{H}^*$ everywhere as follows. By definition $E_1 \equiv 1$; given $E_1, \ldots, E_n$, we take as E_{n+1} any function which is $\mathscr{H}^*$ everywhere and is such that

$$|E_n . F_n . E_{n+1}^{-1} - 1| < \delta_n, \text{ on } \overline{D_1^n} \cup \overline{D_2^n}. \tag{2}$$

The existence of such a function is guaranteed by our Lemma, since $E_n . F_n$ is $\mathscr{H}^*$ on $\overline{D_1^n} \cup \overline{D_2^n}$. Let us also define two other sequences $(\phi_n)_1^\infty$, $(\psi_n)_1^\infty$ as follows :

$$\phi_n = g_n . E_n^{-1} \text{ on } \overline{D_1^n},$$

$$\psi_n = h_n . E_n^{-1} \text{ on } \overline{D_2^n}.$$

Then from (1),

$$E_n . F_n . E_{n+1}^{-1} = \begin{cases} \phi_n^{-1} . \phi_{n+1}, \text{ on } \overline{D_1^n}, \\ \psi_n^{-1} . \psi_{n+1}, \text{ on } \overline{D_2^n}. \end{cases}$$

By iterated application of the first of these results we obtain

$$\phi_{n+p} = \phi_n \prod_{k=n}^{n+p} E_k . F_k . E_{k+1}^{-1}, \text{ on } \overline{D_1^n}, \tag{3}$$

where p is any positive integer. Now from (2) it follows in the usual way that the infinite product $\prod_{k=n}^{\infty} E_k . F_k . E_{k+1}^{-1}$ converges uniformly on $\overline{D_1^n} \cup \overline{D_2^n}$. Since each factor is $\mathscr{H}^*$ on this closed region, it follows by the multiplicative analogue of Weierstrass's theorem (cf. [6, *p.* 219]) that so is its limit, say P_n. From (3) we then see that on $\overline{D_1^n}$, $f_1 = \lim_{p \to \infty} \phi_{n+p}$ exists and equals $\phi_n . P_n$. Clearly f_1 is also $\mathscr{H}^*$ on $\overline{D_1^n}$. As this holds for all positive integers n, we conclude that f_1 is $\mathscr{H}^*$ on D_1.

In exactly the same way we show that $\psi_n \to f_2$, where f_2 is $\mathscr{H}^*$ on D_2. It readily follows that the reciprocal $\psi_n^{-1} \to f_2^{-1}$. Finally for each n we have on $\overline{D_1^n} \cap \overline{D_2^n}$,

$$f = g_n . h_n^{-1} = g_n . E_n^{-1} . E_n . h_n^{-1} = \phi_n . \psi_n^{-1},$$

and given any $z \in D_1 \cap D_2$, we have $z \in \overline{D_1^n} \cap \overline{D_2^n}$, for all sufficiently large n. It follows by letting $n \to \infty$ that on $D_1 \cap D_2$

$$f = f_1 \cdot f_2^{-1}.$$

3. EXTENSIONS. It should be noted that our proof of Theorem I can easily be converted into a proof of the following theorem by everywhere replacing multiplication by addition, reciprocals by negatives, $\mathscr{H}^*$ by $\mathscr{H}$, and the entirely reciprocable functions E_n by ordinary polyinomials P_n.

THEOREM II. *Given D_1, D_2 and D as in Theorem* I *and a complex-valued function f holomorphic on D then f is the sum of two functions f_1 and f_2 such that f_1 is holmorphic on D_1 and f_2 is holomorphic on D_2.*

This additive version of Theorem I is however a very special case of stronger results due to Aronszajn [1]. Moreover when f is assumed to be continuous on the closure of D, this theorem, unlike Theorem I, follows trivially from the strong form of Cauchy's integral representation.

It is also clear that our proof of Theorem I can be made to work for any simply connected regions D_1 and D_2 of the complex-plane whose boundaries meet in two points, and for which $D = D_1 \cap D_2$ is bounded and connected. Extensions are also possible to certain cases when D is unbounded. Thus for the strip we have

THEOREM III. *Theorem* I *remains valid when D is the open strip bounded by parallel lines L_1 and L_2, D_1 the open half-plane containing L_2 and bounded by L_1, and D_2 the open half-plane containing L_1 and bounded by L_2.*

PROOF. We may, without loss of generality, take $L_1 = [y = 1]$ and $L_2 = [y = -1]$, where $z = x + iy$, so that $D = [-1 < y < 1]$. We then repeat our previous proof letting however, for $n \geqslant 2$, D_1^n be the isosceles triangle with vertex

$(1 - 1/n)\, i$, whose sides through this vertex pass through n and $-n$, and whose altitude through this vertex is of length n, and letting D_2^n be the reflection of D_1^n in the x-axis.

The bearing of Theorem III on the factorization problem which has arisen in the filtering and prediction of multiple time series (cf. [9, Ch. IV]), as well as possible multiplicative extensions to the non-commutative case of Aronszajn's results are being studied.

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ON GAUSS'S SUMS

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For integral values of a, x, M let us define the function $F(a, x, M)$ by the relation

$$F(a, x, M) = \sum_{m(\text{mod } M)} \exp \frac{2\pi i}{M}(am^2 + xm),$$

where $\sum_{m(\text{mod } M)}$ indicates summation over a complete set of residues modulo M.

We establish below some properties of the function $F(a, x, M)$.

THEOREM 1. *If* $N = aA^2 + bB^2$ *is prime to* M, *then* $F(a,x,M)\,F(b,y,M) = F(N, Ax + By, M)\,F(Nab, aAy - bBx, M)$.

PROOF. Since N is prime to M, the g.c.d. of A, B is prime to M and so we can find two integers C, D such that

$$AD - BC = 1 \bmod M.$$

In fact, if

$$A = A'g, B = B'g, \qquad \text{g.c.d. of } A', B' = 1,$$

and C', D' be two numbers such that

$$A'D' - B'C' = 1,$$

then

$$AD' - BC' = g$$

so that

$$A(\bar{g}D') - B(\bar{g}C') \equiv 1 \bmod M,$$

where $\bar{g}$ is the inverse of g mod M, and we have only to take

$$C = \bar{g}C', D = \bar{g}D'.$$

Now

$$F(a, x, M)\,F(b, y, M) = \sum_{m_1, m_2 (\text{mod } M)} \exp \frac{2\pi i}{M}(P + Q),$$

where

$$P = am_1^2 + bm_2^2, \; Q = xm_1 + ym_2.$$

Since $AD - BC \equiv 1 \bmod M$, we can always find two integers $u_1, u_2 \bmod M$ such that

$$\left.\begin{aligned} Au_1 + Cu_2 &\equiv m_1 \\ Bu_1 + Du_2 &\equiv m_2 \end{aligned}\right\} \bmod M.$$

In fact we have

$$\left.\begin{aligned} u_1 &\equiv Dm_1 - Cm_2 \\ u_2 &\equiv - Bm_1 + Am_2 \end{aligned}\right\} \bmod M.$$

It is also easily seen that as m_1, m_2 run independently through complete sets of residues mod M, u_1, u_2 also do so. It follows that we may write

$$F(a, x, M)\, F(b, y, M) = \sum_{u_1, u_2 (\bmod M)} \exp \frac{2\pi i}{M}(P + Q),$$

where

$$\begin{aligned} P &= a(Au_1 + Cu_2)^2 + b(Bu_1 + Du_2)^2 \\ &= Nu_1^2 + 2N' u_1 u_2 + N'' u_2^2, \\ Q &= x(Au_1 + Cu_2) + y(Bu_1 + Du_2), \end{aligned}$$

N, N', N'' being given by

$$N = aA^2 + bB^2, \; N' = aAC + bBD, \; N'' = aC^2 + bD^2,$$

so that

$$NN'' - N'^2 = ab(AD - BC)^2 \equiv ab \bmod M.$$

Since N is prime to M by hypothesis, Nu_2 runs through a complete set of residues mod M when u_2 does so. It follows that without loss of generality we may replace u_2 by Nu_2 in the expressions for P and Q so that we have

$$\begin{aligned} P &= N\,[u_1^2 + 2N'u_1 u_2 + NN''u_2^2] \\ &= N\,[(u_1 + N'u_2)^2 + (NN'' - N'^2)\, u_2^2] \\ &\equiv N\,[(u_1 + N'u_2)^2 + abu_2^2] \bmod M, \end{aligned}$$

and

$$Q = (Ax + By)(u_1 + N'u_2) + \{N(Cx + Dy) - N'(Ax + By)\}\, u_2$$
$$= (Ax + By)(u_1 + N'u_2) + (aAy - bBx)(AD - BC)\, u_2$$
$$\equiv (Ax + By)(u_1 + N'u_2) + (aAy - bBx)\, u_2 \bmod M.$$

Again, for a given u_2, since $u_1 + N'u_2$ runs through a complete set of residues mod M when u_1 does so, we may, without loss of generality, replace $u_1 + N'u_2$ by u_1 in the expressions for P and Q and finally get

$$\left.\begin{aligned} P &\equiv N\,(u_1^2 + abu_2^2) \\ Q &\equiv (Ax + By)\, u_1 + (aAy - bBx)\, u_2 \end{aligned}\right\} \bmod M,$$

so that we have

$$F(a, x, M).\ F(b, y, M) = \sum_{u_1,\, u_2 (\mathrm{mod}\, M)} \exp \frac{2\pi i}{M} \Big(N(u_1^2 + abu_2^2) + $$
$$+ (Ax + By)\, u_1 + (aAy - bBx) u_2 \Big)$$

$$= \sum_{u_1 (\mathrm{mod}\, M)} \exp \frac{2\pi i}{M} \{ Nu_1^2 + (Ax + By)\, u_1 \} \times$$
$$\times \sum_{u_2 (\mathrm{mod}\, M)} \exp \frac{2\pi i}{M} \Big(Nabu_2^2 + (aAy - bBx)\, u_2 \Big)$$
$$= F(N, Ax + By, M)\ F(Nab, aAy - bBx, M).$$

THEOREM 2. *If* $N = X^2 + aY^2 + bZ^2 + abW^2$ *is prime to* M, *then*

$$F(1, x, M)\ F(a, y, M)\ F(b, z, M)\ F(ab, w, M)$$
$$= F(N, x', M)\ F(aN, y', M)\ F(bN, z', M)\ F(abN, w', M),$$

where

$$x' = Xx - Yy - Zz - Ww,$$
$$y' = aYx + Xy - aWz + Zw,$$
$$z' = bZx + bWy + Xz - Yw,$$
$$w' = abWx - bZy + aYz + Xw.$$

PROOF. We have

$$F(1, x, M)\, F(a, y, M)\, F(b, z, M)\, F(ab, w, M)$$

$$= \sum_{m_1, m_2, m_3, m_4 \pmod M} \exp \frac{2\pi i}{M}(P+Q),$$

where

$$P = m_1^2 + am_2^2 + bm_3^2 + abm_4^2; \quad Q = m_1x + m_2y + m_3z + m_4w.$$

Since N is prime to M we may replace m_i by Nm_i, $(i = 1, 2, 3, 4)$, in the expressions for P and Q without loss of generality. Write

$$P = N^2(m_1^2 + am_2^2 + bm_3^2 + abm_4^2)$$

$$= N(X^2 + aY^2 + bZ^2 + abW^2)(m_1^2 + am_2^2 + bm_3^2 + abm_4^2)$$

$$= N(u_1^2 + au_2^2 + bu_3^2 + abu_4^2),$$

where, by a well-known property of the norm of quaternions, we have, in matrix notation,

$$(u_1, u_2, u_3, u_4) = (m_1, m_2, m_3, m_4)\,\mathcal{H},$$

$\mathcal{H}$ being the matrix

$$\begin{pmatrix} X & Y & Z & W \\ -aY & X & aW & -Z \\ -bZ & -bW & X & Y \\ -abW & bZ & -aY & X \end{pmatrix};$$

so that we have also

$$(u_1, u_2, u_3, u_4)\,\mathcal{H}^* = (Nm_1, Nm_2, Nm_4, Nm_4),$$

$\mathcal{H}^*$ being the matrix

$$\begin{pmatrix} X & -Y & -Z & -W \\ aY & X & -aW & Z \\ bZ & bW & X & -Y \\ abW & -bZ & aY & X \end{pmatrix},$$

since it may readily be seen that

$$\mathcal{H}.\mathcal{H}^* = N\mathcal{I},$$

$\mathcal{I}$ being the identity matrix; and

$$Q = N\,(m_1x + m_2y + m_3z + m_4w)$$
$$= (u_1, u_2, u_3, u_4)\,\mathcal{H}^* \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$
$$= (u_1, u_2, u_3, u_4) \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix}$$
$$= u_1x' + u_2y' + u_3z' + u_4w'.$$

We next show that as m_1, m_2, m_3, m_4 run independently through residue systems mod M, u_1, u_2, u_3, u_4 also do so. For this, it is enough to prove that if (m_1', m_2', m_3', m_4') and $(m_1'', m_2'', m_3'', m_4'')$ are two sets of values of (m_1, m_2, m_3, m_4) and (u_1', u_2', u_3', u_4') and $(u_1'', u_2'', u_3'', u_4'')$ are the corresponding values of (u_1, u_2, u_3, u_4) then

$$(u_1', u_2', u_3', u_4') \equiv (u_1'', u_2'', u_3'', u_4'') \bmod M$$

if and only if

$$(m_1', m_2', m_3', m_4') \equiv (m_1'', m_2'', m_3'', m_4'') \bmod M.$$

The sufficiency of the condition follows readily from the relation

$$\begin{aligned}&(u_1' - u_1'',\ u_2' - u_2'',\ u_3' - u_3'',\ u_4' - u_4'')\\ &= (m_1' - m_1'',\ m_2' - m_2'',\ m_3' - m_3'',\ m_4' - m_4'')\,\mathcal{H}\\ &\equiv (0, 0, 0, 0) \bmod M,\end{aligned}$$

and the necessity from the relation

$$\begin{aligned}(0, 0, 0, 0) &\equiv (u_1' - u_1'',\ u_2' - u_2'',\ u_3' - u_3'',\ u_4' - u_4'')\,\mathcal{H}^*\\ &= N\,(m_1'-m_1'', m_2'-m_2'', m_3'-m_3'', m_4'-m_4'') \bmod M,\end{aligned}$$

combined with the fact that the factor N may be cancelled as it is prime to M.

Thus we have finally

$$F(1, x, M)\ F(a, y, M)\ F(b, z, M)\ F(ab, w, M)$$
$$= \sum_{u_1, u_2, u_3, u_4, \,(\mathrm{mod}\ M)} \exp \frac{2\pi i}{M}(P + Q),$$

where

$$P = N\ (u_1^2 + au_2^2 + bu_3^2 + abu_4^2)$$

and

$$Q = u_1x + u_2y' + u_3z' + u_4w',$$

from which the required result follows.

We finally observe that results similar to those of Theorem 2 can be obtained for the products of 8 and 16 Gauss's sums because of the availability of tools similar to those used in the proof of that theorem in those cases also.

It may also be observed that Theorem 1 is analogous to the generalization of Schröter's theorem for products of ϑ-functions obtained by me earlier [1]. Results analogous to Theorem 2 also exist for ϑ-functions. These have some special points of interest as they include Jacobi's fundamental formulae as a special case and are therefore being given separately.

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LINEAR DIFFERENCE EQUATIONS ASSOCIATED WITH CERTAIN SPECIAL FUNCTIONS

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1. The functional equation

$$(n-1)\,f_{n+1}(z)+(n+1)\,f_{n-1}(z)-\frac{2}{z}\,(n^2-1)\,f_n(z)=0 \tag{1}$$

can be transformed into the form

$$\phi_{n+1}(z)+\phi_{n-1}(z)=\frac{2n}{z}\,\phi_n(z) \tag{2}$$

by the substitution

$$f_n(z)=n\,\phi_n(z). \tag{3}$$

Because (2) is satisfied by each of the functions

$$J_n(z),\ Y_n(z),\ Y^{(n)}(z)\ H_n^{(1)}(z) \text{ and } H_n^{(2)}(z),$$

the general solution of (2) is of the form

$$\phi_n(z)=\alpha_n(z)\,g(z)+\beta_n(z)\,h(z), \tag{4}$$

where $g(z)$ and $h(z)$ are arbitrary functions, and $\alpha_n(z)$, $\beta_n(z)$ are any two of the functions mentioned above. The solution of (1) is then given by (3) and (4). This solution will form the complementary function of the difference equation

$$(n-1)f_{n+1}(z)+(n+1)f_{n+1}(z)-\frac{2}{z}(n^2-1)\,f_n(z)=\frac{2n}{z}\sin^2\frac{n\pi}{2}. \tag{I}$$

A particular solution of (I) being the Neumann polynomial $O_n(z)$, it follows that the complete solution of (I) is

$$f_n(z)=n\,[\alpha_n(z)\,g(z)+\beta_n(z)\,h(z)]+O_n(z) \tag{6}$$

where $\alpha_n(z)$, $\beta_n(z)$, $g(z)$ and $h(z)$ bear the meanings ascribed to them above.

2. We proceed to discuss the common solutions of

$$(n-1)f_{n+1}(z)+(n+1)f_{n-1}(z)-\frac{2}{z}(n^2-1)f_n(z)=\frac{2n}{z}\sin^2\frac{n\pi}{2}, \quad \text{(I)}$$

and

$$f_{n-1}(z)-f_{n+1}(z)=2f_n'(z). \quad \text{(II)}$$

We shall set $\alpha_n(z)=J_n(z)$ and $\beta_n(z)=Y_n(z)$ in (6). (6) will then satisfy (II), if

$$\begin{aligned} g(z)\,&[(n-1)\,J_{n-1}(z)-(n+1)\,J_{n+1}(z)-2n\,J_n'(z)]+\\ &+h(z)\,[(n-1)\,Y_{n-1}(z)-(n+1)\,Y_{n+1}(z)-2n\,Y_n'(z)]+\\ &+[O_{n-1}(z)-O_{n+1}(z)-2\,O_n'(z)]\\ &\qquad=2n\,[J_n(z)\,g'(z)+Y_n(z)\,h'(z)]. \end{aligned}$$

This simplifes to

$$\begin{aligned} g(z)\,[J_{n-1}(z)+J_{n+1}(z)]+h(z)\,[Y_{n-1}(z)+Y_{n+1}(z)]+\\ +2n\,[J_n(z)\,g'(z)+Y_n(z)\,h'(z)]=0, \end{aligned} \quad (7)$$

since each of the functions $J_n(z)$, $Y_n(z)$ and $O_n(z)$ satisfies the relation (II).

Using the recurrence formulae for $J_n(z)$ and $Y_n(z)$, (7) further reduces to the form

$$J_n(z)\,\gamma(z)+Y_n(z)\,\delta(z)=0, \quad (8)$$

where

$$\gamma(z)\equiv g(z)+zg'(z)\,;\quad \delta(z)\equiv h(z)+zh'(z). \quad (9).$$

We show that $\gamma(z)$ and $\delta(z)$ must vanish identically, if (8) holds for all values of z and for all positive integral values of n. We first observe that the vanishing of one of them entails that of the other, since $Y_n(z)$ or $J_n(z)$ is not identically zero. Suppose now that while (8) is true, neither $\gamma(z)$ nor $\delta(z)$ vanishes identically. This would show that $J_n(z)/\,Y_n(z)$ is independent of n, for all values of z.

This being impossible, it follows that $\gamma(z)\equiv 0$, and $\delta(z)\equiv 0$. Integrating (9), we obtain

$$g(z)=Az^{-1},\; h(z)=Bz^{-1},$$

where A and B are arbitrary constants (independent of n). The required common solution of (I) and (II) is therefore given by

$$f_n(z) = nz^{-1}\,[AJ_n(z) + BY_n(z)] + O_n(z). \tag{10}$$

We may note here that the differential equation

$$\frac{d^2w}{dz^2} + \frac{3}{z}\frac{dw}{dz} + \left(1 - \frac{n^2-1}{z^2}\right)w = \frac{1}{z}\cos^2\frac{n\pi}{2} + \frac{n}{z^2}\sin^2\frac{n\pi}{2} \tag{III}$$

has its general solution given by the same form (10), except that A and B need not necessarily be independent of n. Hence, every common solution of (I) and (II) satisfies (III), but not conversely. It is also easy directly to verify that if $f_n(z)$ is a function which satisfies (I) and (II), it satisfies (III) as well.

3. We discuss similarly the simultaneous equations

$$f_{n-1}(z) - f_{n+1}(z) = 2f_n'(z), \tag{II}$$

and

$$f_{n+1}(z) + f_{n-1}(z) - \frac{2n}{z}f_n(z) = \frac{4}{z}\cos^2\frac{n\pi}{2}. \tag{IV}$$

The complete solution of (IV) is known to be

$$f_n(z) = J_n(z)\,g(z) + Y_n(z)\,h(z) + S_n(z), \tag{11}$$

where $S_n(z)$ is Schläfli's polynomial, and $g(z)$, $h(z)$ are arbitrary functions. (11) will satisfy (II) if

$$J_n(z)\,g'(z) + Y_n(z)\,h'(z) = 0.$$

Repeating the argument of § 2, it follows that $g(z)$ and $h(z)$ are arbitrary constants, *independent of* n. The common solution of (II) and (IV) is then given by

$$f_n(z) = aJ_n(z) + bY_n(z) + S_n(z), \tag{12}$$

where a and b are arbitrary constants, independent of n. It may be directly verified as before that any common solution of (II) and (IV) satisfies the differential equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{n^2}{z^2}\right)w = \frac{2}{z}\sin^2\frac{n\pi}{2} + \frac{2n}{z^2}\cos^2\frac{n\pi}{2}. \tag{V}$$

But the general solution of (V) is given by (12), where a and b are arbitrary constants, not necessarily independent of n.

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ON THE HANKEL TRANSFORMATION OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

By DINESH CHANDRA

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1. Hankel transformations of various functions have been given from time to time. Recently Hari Shankar [1] obtained many of these from the generalized hypergeometric function ${}_2F_2\,[\alpha, \beta\,;\,\gamma, \delta\,;\,-x]$. The object of this note is to deduce the transformation of the most general function

$$ {}_nF_n\,[a_1, a_2, \ldots, a_n\,;\,\rho_1, \rho_2, \ldots, \rho_n\,;\,-x]. $$

All the results deduced by Hari Shankar follow as particular cases from the results obtained by me.

2. Let us write

$$ \phi(x) = \frac{x^{2\rho_n-\nu-3/2}}{\Gamma(\rho_n)\,2^{\rho_n-1}}\,{}_nF_n[a_1, a_2, \ldots, a_n\,;\,\rho_1, \rho_2, \ldots, \rho_n;\,-\tfrac{1}{4}x^2] $$

and

$$ \psi(x) = \frac{\Gamma(\rho_1)\,\Gamma(\rho_2)\ldots\Gamma(\rho_{n-1})}{\Gamma(a_1)\,\Gamma(a_2)\ldots\Gamma(a_n)} \sum_{r=1}^{n} \frac{\prod\limits_{t=1}^{n}{}' \,\Gamma(a_t - a_r)}{\prod\limits_{s=1}^{n-1} \Gamma(\rho_s - a_r)}\,\Gamma(a_r)\,\times $$

$$ \times\,\frac{x^{2a_r+\nu-2\rho_n+1/2}}{2^{a_r+\nu-\rho_n}\,\Gamma(a_r+\nu-\rho_n+1)}\,\times $$

$$ \times\,{}_nF_n\,[a_r, a_r-\rho_1+1, \ldots, a_r-\rho_{n-1}+1\,;\,-\tfrac{1}{4}x^2, a_r-a_1+1, \ldots,{}^* \ldots, a_r-a_n+1, a_r+\nu-\rho_n+1] $$

where the dash indicates that $t \neq r$ and the asterisk indicates that the expression $a_r - a_r + 1$ is omitted.

Then $\phi(x)$ and $\psi(x)$ are Hankel transforms of each other of order ν, provided that [5]

$R(\rho_n) > 0$, $R(2\,a_r - 2\,\rho_n + \nu + \frac{3}{2}) > 0$, $R(a_r - \rho_n + \nu + 1) > 0$, $R(\rho_n) > R(\frac{1}{2}\nu + \frac{1}{4})$ and $R(\rho_n - \rho_s) > R(\frac{1}{2}\nu - \frac{3}{4})$; $(r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, n-1)$.

3. We shall establish the theorem with the help of operational calculus. If

$$f(p) = p \int_0^\infty e^{-px}\, h(x)\, dx,$$

then

$$f(p) \doteqdot h(x).$$

It is known that [4]

$$\frac{1}{p^{\nu-1}} f\left(\frac{1}{p}\right) \doteqdot \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\nu} J_\nu(2\sqrt{xt})\, h(t)\, dt, \tag{1}$$

assuming that the integral converges.

Let us assume that

$$h(x) = \frac{x^{\rho_n - 1}}{\Gamma(\rho_n)}\, {}_nF_n[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_n; -x]\, dx.$$

Then

$$\begin{aligned} f(p) &= \frac{p}{\Gamma(\rho_n)} \int_0^\infty e^{-px}\, x^{\rho_n - 1}\, {}_nF_n[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_n; -x]\, dx \\ &= \frac{1}{p^{\rho_n - 1}}\, {}_nF_{n-1}[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_{n-1}; -1/p], \end{aligned}$$

term-by-term integration being permissible. Hence

$$\begin{aligned} &p^{1-\rho_n}\, {}_nF_{n-1}[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_{n-1}; -1/p] \\ &\doteqdot \frac{1}{\Gamma(\rho_n)} x^{\rho_n - 1}\, {}_nF_n[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_n; -x]. \end{aligned}$$

We therefore have

$$\begin{aligned} &\frac{\Gamma(\rho_n)}{p^{\nu-\rho_n}}\, {}_nF_{n-1}[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_{n-1}; -p] \\ &= x^{\frac{1}{2}\nu} \int_0^\infty t^{\rho_n - \frac{1}{2}\nu - 1} J_\nu(2\sqrt{xt})\, {}_nF_n[a_1, a_2, \ldots, a_n; \rho_1, \rho_2, \ldots, \rho_n; -t]\, dt. \end{aligned} \tag{2}$$

Now it has been shown by MacRobert and Thomae [2, 3] that

$$\sum_{r=1}^{n} \frac{\prod_{t=1}^{n}{}' \Gamma(a_t - a_r)}{\prod_{s=1}^{n-1} \Gamma(\rho_s - a_r)} \Gamma(a_r) x^{a_r} {}_nF_{n-1}\left[\begin{matrix} a_r, a_r - \rho_1 + 1, \ldots, a_r - \rho_{n-1} + 1; \\ a_r - a_1 + 1, \ldots * \ldots, a_r - a_n + 1 ; -x \end{matrix}\right]$$

$$(=) \frac{\Gamma(a_1) \ldots \Gamma(a_n)}{\Gamma(\rho_1) \ldots \Gamma(\rho_{n-1})} {}_nF_{n-1}[a_1, \ldots, a_n; \rho_1, \ldots, \rho_{n-1}; -1/x] \quad (3)$$

and $-\pi < \text{amp}\ x < \pi$, the function being made uniform by a cross-cut from the origin through -1 and $-\infty$, provides the analytic continuation of the left side valid for $|x| < 1$ into the region for which $|x| > 1$. The symbol $(=)$ shows that the expressions on the two sides represent the same function each in its own domain. Since $p > 1$,

$$\frac{1}{p^{\nu-\varrho_n}} {}_nF_{n-1}[a_1, \ldots, a_n; \rho_1, \ldots, \rho_{n-1}; -p]$$

$$= \frac{1}{p^{\nu-\varrho_n}} \frac{\Gamma(\rho_1) \ldots \Gamma(\rho_{n-1})}{\Gamma(a_1) \ldots \Gamma(a_n)} \sum_{r=1}^{n} \frac{\prod_{t=1}^{n}{}' \Gamma(a_t - a_r)}{\prod_{s=1}^{n-1} \Gamma(\rho_s - a_r)} \cdot \frac{\Gamma(a_r)}{p^{a_r}} {}_nF_{n-1}[a^r, a_r - \rho_1 + 1,$$

$$\ldots, a_r - \rho_{n-1} + 1, a_r - a_1 + 1, \ldots * \ldots, a_r - a_n + 1, -1/p]$$

and

$$\frac{1}{p^{a_r+\nu-\varrho_n}} {}_nF_{n-1}[a_r, a_r - \rho_1 + 1, \ldots, a_r - \rho_{n-1} + 1, a_r - a_1 + 1, \ldots * \ldots, a_r - a_n + 1, -1/p]$$

$$\doteqdot \frac{t^{a_r+\nu-\varrho_n}}{\Gamma(a_r + \nu - \rho_n + 1)} {}_nF_n[a_r, a_r - \rho_1 + 1, \ldots, a_r - \rho_{n-1} + 1, a_r - a_1 + 1, \ldots * \ldots, a_r - a_n + 1, a_r + \nu - \rho_n + 1; -t]. \quad (4)$$

Hence

$$\frac{1}{p^{\nu-\varrho_n}} {}_nF_{n-1}[a_1, \ldots, a_n, \rho_1, \ldots, \rho_{n-1}, -p]$$

$$\doteqdot \frac{\Gamma(\rho_1) \ldots \Gamma(\rho_{n-1})}{\Gamma(a_1) \ldots \Gamma(a_n)} \sum_{r=1}^{n} \frac{\prod_{t=1}^{n}{}' \Gamma(a_t - a_r) \Gamma(a_r) t^{a_r+\nu-\varrho_n}}{\prod_{s=1}^{n-1} \Gamma(\rho_s - a_r) \Gamma(a_r + \nu - \rho_n + 1)} {}_nF_n[a_r,$$

$$a_r - \rho_1 + 1, \ldots, a_r - \rho_{n-1} + 1; a_r - a_1 + 1, \ldots * \ldots, a_r - a_n + 1, a_r + \nu - \rho_n + 1; -t]. \quad (5)$$

Hence taking into account the relations (2) and (5), we get

$$x^{\frac{1}{2}\nu}\int_0^\infty t^{\varrho_n-\frac{1}{2}\nu-1}\,J_\nu(2\sqrt{xt})\;{}_nF_n\,[a_1, a_2, \ldots, a_n;\, \rho_1, \rho_2, \ldots, \rho_n;\, -t]\,dt$$

$$=\frac{\Gamma(\rho_1)\ldots\Gamma(\rho_n)}{\Gamma(a_1)\ldots\Gamma(a_n)}\sum_{r=1}^{n}\frac{\prod\limits_{t=1}^{n}{}'\,\Gamma(a_t-a_r)\,\Gamma(a_r)\,x^{a_r+\nu-\varrho_n}}{\prod\limits_{s=1}^{n}\Gamma(\rho_s-a_r)\,\Gamma(a_r+\nu-\rho_n+1)}\times$$

$$\times\;{}_nF_n\,[a_r, a_s-\rho_1+1, \ldots, a_r-\rho_{n-1}+1, a_r-a_1+1, \ldots{}^*\ldots,$$

$$a_r-a_n+1, a_r+\nu-\rho_n+1;\, -x]. \qquad (6)$$

Writing $\frac{1}{2}x^2$ for x and $\frac{1}{2}t^2$ for t, we get after a slight modification,

$$\frac{1}{2^{\varrho_n-1}}\int_0^\infty \sqrt{xt}\;t^{2\varrho_n-\nu-3/2}\,J_\nu\,(xt)\;{}_nF_n\,[a_1, \ldots, a_n;\, \rho_1, \rho_2, \ldots \rho_n;\, -\tfrac{1}{2}t^2]\,dt$$

$$=\frac{\Gamma(\rho_1)\ldots\Gamma(\rho_n)}{\Gamma(a_1)\ldots\Gamma(a_n)}\sum_{r=1}^{n}\frac{\prod\limits_{t=1}^{n}{}'\;\Gamma(a_t-a_r)\,\Gamma(a_r)\,x^{2a_r+\nu-2a_n+1/2}}{\prod\limits_{s=1}^{n-1}\Gamma(\rho_s-a_r)\,\Gamma a_r+\nu-\rho_n+1)\,2^{a_r+\nu-\varrho_n}}\times$$

$$\times\;{}_nF_n\,[a_r, a_r-\rho_1+1, \ldots, a_r-\rho_{n-1}+1;\, a_r-a_1+1, \ldots,$$

$$a_r-a_{n+1}, a_r+\nu-\rho_{n+1};\, -\tfrac{1}{2}x^2], \qquad (7)$$

i.e.
$$\int_0^\infty (xt)^{1/2}\,J_\nu(xt)\,\phi(t)\,dt=\psi(x). \qquad (7')$$

4. Let $m_1, m_2, \ldots, m_{n-1}$ be $n-1$ positive integers. Let us write $\frac{1}{2}(a_1+m_1+1)$ for $a_2, \ldots, \frac{1}{2}(a_1+m_{n-1}+1)$ for a_n and $\frac{1}{2}(a_1-m_1+1)$ for ρ_1, $\frac{1}{2}(a_1-m_2+1)$ for $\rho_2, \ldots,$ and $\frac{1}{2}(a_1+\nu+1)$ for ρ_n. We easily prove that

$$\int_0^\infty \sqrt{xt}\,J_\nu(xt)t^{a_1-1/2}\;{}_nF_n[a_1, \tfrac{1}{2}(a_1+m_1+1), \ldots, \tfrac{1}{2}(a_1+m_{n-1}+1;$$

$$-\tfrac{1}{2}t^2, \tfrac{1}{2}(a_1-m_1+1), \ldots, \tfrac{1}{2}(a_1-m_{n-1}+1), \tfrac{1}{2}(a_1+\nu+1)\,]\,dt$$

$$=(-1)^{m_1+\ldots+m_{n-1}}\,x^{a_1-1/2}\;{}_nF_n\,[a_1, \tfrac{1}{2}(a_1+m_1+1), \ldots,$$

$$\tfrac{1}{2}(a_1+m_{n-1}+1);\, -\tfrac{1}{2}x^2;\, \tfrac{1}{2}(a_1-m_1+1), \tfrac{1}{2}(a_1-m_{n-1}+1),$$

$$\tfrac{1}{2}(a_1+\nu+1)\,], \qquad (10)$$

showing that the function

$$x^{a_1-1/2}{}_nF_n[a_1, \tfrac{1}{2}(a_1+m_1+1), \ldots, \tfrac{1}{2}(a_1+m_{n-1}+1);\, -\tfrac{1}{2}x^2;$$

$$\tfrac{1}{2}(a_1-m_1+1), \ldots, \tfrac{1}{2}(a_1-m_{n-1}+1), \tfrac{1}{2}(a_1+\nu+1)\,]$$

is self-reciprocal or skew-self reciprocal according as $m_1 + m_2 + \dots + m_{n-1}$ is an even or an odd integer.

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ON A ONE-SIDED TAUBERIAN THEOREM

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1. Introduction. The principal object of this note is to generalize a one-sided Tauberian theorem of Karamata [3] for the Abel transform of a series. Implicit in the proof of the generalization there is a similar generalization of another one-sided Tauberian theorem, stated without proof by Vijayaraghavan [7, Theorem 8] for the Abel transform. These generalizations, in the order mentioned, appear below as Theorems A, a; they concern, instead of the Abel transform, a transform Φ in which the kernel $\phi(u)$ satisfies the following conditions.

(C) For $u \geqslant 0$, $\phi(u)$ is positive, twice differentiable, and $\phi'(u) < 0$; further

$$\phi(0) = 1, \int^{\infty} \frac{\phi(u)}{u}\, du \text{ is convergent.}$$

We can state the generalizations of the theorems of Karamata and Vijayaraghavan for the Φ-transform as follows.

THEOREM A. *Suppose that $a(u)$ is a real function, integrable over each finite interval $0 \leqslant u \leqslant u_0$, and such that*

$$a(u) = 0 \textit{ for } 0 \leqslant u < 1,\ u\, a(u) \geqslant -W,\ W > 0, \textit{ for } u \geqslant 1. \quad (1)$$

Suppose that $\phi(u)$ satisfies conditions (C) *and*

$$\Phi(t) = \int_0^{\infty} \phi(ut)\, a(u)\, du \quad \textit{exists for } t > 0,$$

$$\Phi(t) \leqslant M. \quad (2)$$

Then $A(h)$ defined by

$$A(h) = \int_0^h a(u)\, du \equiv \int_1^h a(u)\, du \quad (3)$$

satisfies the inequality

$$A(h) < W \log[-\phi'(0) \log h] + M + W(1-\gamma) + o(1),\ h \to \infty, \quad (4)$$

where γ denotes the following limit which obviously exists:

$$\gamma = \lim_{t \to +0} \left[-\log t - \int_t^\infty \frac{\phi(u)}{u} du \right]. \tag{5}$$

THEOREM a. *If, in Theorem* A, *we replace the hypothesis* (2) *by*

$$\{\Phi(t) + W \log \log 1/t\} \to -\infty \text{ as } t \to +0, \tag{6}$$

and make no other change, the conclusion (4) *will be changed to*

$$\lim_{h \to \infty} A(h) = -\infty. \tag{7}$$

Theorem A may be viewed as a one-sided companion of the following two-sided theorem which I have obtained elsewhere [4] by way of generalizing for the Φ-transform a theorem of Agnew and Hadwiger for the Abel transform [1, pp. 110, 114] independently of a more comprehensive result of Delange [2, Théorème 3] in the same direction.

THEOREM B. *Suppose that $a(u)$ is integrable over each finite interval $0 \leqslant u \leqslant u_0$ and such that*

$$\limsup_{u \to \infty} |u\, a(u)| = W.$$

Suppose that $\phi(u)$ satisfies conditions (C) *and*

$$\Phi(t) = \int_0^\infty \phi(ut)\, a(u)\, du \quad \textit{exists for } t > 0.$$

Then, defining $A(h)$ as in (3), *we get*

$$\limsup_{t \to +0} \left| A\left(\frac{\phi^{-1}(2^{-1})}{t}\right) - \Phi(t) \right| \leqslant W\tau,$$

where

$$\tau = \int_0^{\phi^{-1}(1/2)} \frac{1 - \phi(u)}{u} du + \int_{\phi^{-1}(1/2)}^\infty \frac{\phi(u)}{u} du$$

$$= \gamma + \log \phi^{-1}(2^{-1}) + 2 \int_{\phi^{-1}(1/2)}^\infty \frac{\phi(u)}{u} du,$$

γ being the constant defined in (5).

It is obvious that Theorem B includes the following two-sided counterpart of Theorem a, which in turn includes a theorem of Vijayaraghavan [7, Theorem 1].

THEOREM b. *If, in Theorem* B, *we insert the additional hypothesis*

$$\lim_{t \to +0} \Phi(t) = s, \text{ where } |s| = \infty,$$

the conclusion will assume the more definite form

$$\lim_{h \to \infty} A(h) = s.$$

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2. Remarks on the theorems of § 1. The four theorems of § 1 are applicable to the following kernels:

$\phi(u) = e^{-u},\ u \geqslant 0$ (Laplace-Abel kernel);

$\phi(u) = (1 + u)^{-k},\ u \geqslant 0,\ k > 0$ (Stieltjes kernel);

$\phi(u) = u/(e^{u} - 1),\ u > 0;\ \phi(0) = 1$ (Lambert kernel).

Vijayaraghavan has said, of his versions of Theorems a, b, in which $\phi(u) = e^{-u}$, that they bring out a distinction between the Tauberian hypotheses

$$|u\,a(u)| \leqslant W, \quad u\,a(u) \geqslant -W,$$

absent in the case where $\lim \Phi(t)$ is finite as $t \to +0$, considered by Hardy and Littlewood. We can now say that not only Theorems a and b, but also the more general Theorems A and B, are different in kind from the Hardy-Littlewood class of Tauberian theorems for the three kernels mentioned above, which, from their discussion appearing elsewhere [6, Theorem 2 and its corollaries], are seen to strike the deeper level of the stream of ideas of Wiener and Pitt [5, Theorem 1; 6, Lemma 3].

It may be observed incidentally that, in contrast with the Hardy-Littlewood theorems, Ramaswami's Tauberian oscillation theorems for our kernels may be derived, as I have shown

elsewhere [6, Theorem 1 and its corollaries], from an idea [4, Corollary 1.3; 6, Lemma 5] which issues from Theorem B in a slightly more general form [4, Theorem 1].

Karamata has shown [3], in the case $\phi(u) = e^{-u}$, that Theorem A, like Theorem B, is the best of its kind; more precisely that the function $W \log\log h$ in the right-hand member of (4) cannot be replaced by a function of h which tends to infinity less rapidly as $h \to \infty$. The proof of Theorem A which follows offers a natural explanation of this fact.

3. Proofs of Theorems A, a.

Proof of Theorem A. We start with a preliminary inequality for $A(h)$, $h > 1$, defined by (3). The inequality is obtained using first (2), then (1), and letting $t \to +0$.

$$M \geqslant \int_0^\infty \phi(ut)\, a(u)\, du =$$

$$\geqslant \int_1^h \phi(ut)\, a(u)\, du + \int_h^\infty \phi(ut)\, a(u)\, du$$

$$\geqslant \int_1^h \phi(ut)\, a(u)\, du - W \int_h^\infty \frac{\phi(ut)}{u}\, du =$$

$$\geqslant \int_1^h \phi(ut) \left[a(u) + \frac{W}{u} \right] du - W \int_1^\infty \frac{\phi(ut)}{u}\, du$$

$$\geqslant \phi(ht) \int_1^h \left[a(u) + \frac{W}{u} \right] du - W \int_t^\infty \frac{\phi(u)}{u}\, du =$$

$$\geqslant \phi(ht)\, A(h) + W\, \phi(ht) \log ht + W\, [1 - \phi(ht)] \log t - \\ - W \left[\log t + \int_t^\infty \frac{\phi(u)}{u}\, du \right] =$$

$$\geqslant \phi(ht)\, A(h) + W\, \phi(ht) \log ht + W\, [1 - \phi(ht)] \log t + \\ + W \gamma + o(1), \quad t \to +0,$$

where γ is defined by (5). Consequently

$$A(h) \leqslant \frac{M - W\gamma + o(1)}{\phi(ht)} - W \log (ht) + W \left[\frac{\phi(ht) - 1}{\phi(ht)} \right] \log t, \qquad t \to +0. \quad (8)$$

Corresponding to every arbitrarily large $h > 0$, we can choose $t > 0$ so small that ht is bounded and therefore the term $o(1)/\phi(ht)$ on the right side of (8) is negligible. If our choice of t is also such that the right-hand member of (8), with the term $o(1)/\phi(ht)$ omitted, has the least value in terms of h, that value will clearly be the best majorante for $A(h)$ as $h \to \infty$. Therefore we are led to seek the minimum value, for a given h, of the function

$$f(t) = \frac{M - W\gamma - W \log t}{\phi(ht)} - W \log h. \quad (9)$$

The minimum values in question correspond to the values of t which satisfy the conditions $f'(t) = 0, f''(t) > 0$. The former condition gives at once

$$-\frac{M - W\gamma - W \log t}{[\phi(ht)]^2} \phi'(ht)h - \frac{W}{\phi(ht)t} = 0,$$

or

$$[W \log t - (M - W\gamma)]\, ht \frac{\phi'(ht)}{\phi(ht)} = W. \quad (10)$$

Supposing (as we may) that ht is small as well as t, we find, from (10), that t is to be chosen so as to satisfy the condition

$$W \log t.\, ht. \frac{\phi'(0)}{\phi(0)} \sim W, \text{ or } h \sim \frac{1}{\phi'(0)\, t \log t}, \; h \to \infty. \quad (11)$$

Furthermore, when t satisfies (10), $f''(t) > 0$. For, from (9),

$$f''(t) = \frac{d}{dt}\left[- \frac{1}{\{\phi(ht)\}^2} \{(M - W\gamma - W \log t)\, \phi'(ht)h + \frac{W}{t}\, \phi(ht)\} \right].$$

When t is chosen so as to satisfy (10), the last expression for $f''(t)$ reduces to

$$f''(t) = \frac{Wh}{t\, \phi(ht)} \left[\frac{\phi''(ht)}{\phi'(ht)} + \frac{1}{ht} \right]. \quad (12)$$

The sign of the expression within square brackets in (12) is determined from the fact that

$$\frac{ht\,\phi''(ht)+\phi'(ht)}{ht\,\phi'(ht)}=\frac{\phi'(0)+O(ht)}{\phi'(0)ht+O(h^2t^2)}\to\infty,\quad \text{as}\quad ht\to+0.$$

Remembering that, in (10), ht is small, we now conclude from (12) that t satisfying (10) makes $f''(t)>0$. Hence, for arbitrarily large h, the best form of (8), i.e. the form with min $f(t)$ under the assumed conditions, is obtained by using (11) in (8) and writing (8) in the form

$$A(h)\leqslant\frac{M-W\gamma+o(1)}{\phi(ht)}+W\log\,[\phi'(0)\log t]+o(1)+$$
$$+W\left[\frac{\phi(ht)-1}{\phi(ht)}\right]\log t, \tag{13}$$

where, on account of (11),

$$W\left[\frac{\phi(ht)-1}{\phi(ht)}\right]\log t\sim W\,ht\,\phi'(0)\log t\sim W. \tag{14}$$

Using (14) in (13), we obtain, when we let $t\to+0$, and define h by (11),

$$A(h)<M-W\gamma+W\log\,[\phi'(0)\log t]+W+o(1). \tag{15}$$

To complete the proof we have to rewrite the inequality (15) in terms of h alone. This is easily done since, when $t\to+0$, (11) gives

$$\log h=\log\left[\frac{1}{\phi'(0)\;t\log t}\right]+o(1)$$
$$=-\log\,[-\phi'(0)]+\log\frac{1}{t}-\log\log\frac{1}{t}+o(1), \tag{16}$$

i.e.

$$\log\log h=\log\log\frac{1}{t}+o(1).$$

Employing the last relation in (15), we are immediately led to the desired conclusion (4) in the form

$$A(h)<M-W\gamma+W\log\,[-\phi'(0)]+W\log\log h+W+o(1),$$
$$h\to\infty.$$

PROOF OF THEOREM a. Given any large positive G, (6) implies that for all sufficiently small t,

$$\Phi(t) < -G - W \log\log \frac{1}{t}. \tag{17}$$

Use (17) instead of (2) and, in the inequality which results in place of (8), suppose that $t \to +0$ and h satisfies (11) with its implications (14), (16). Then

$$A(h) < \left[-G - W\log\log\frac{1}{t} - W\gamma + o(1)\right][1 + O(ht)] +$$

$$+ W\log[-\phi'(0)] + W\log\log\frac{1}{t} + W + o(1) =$$

$$< \left[-G - W\log\log\frac{1}{t} - W\gamma\right]\left[1 + O\left(\frac{1}{-\phi'(0)\log(1/t)}\right)\right]$$

$$+ W\log[-\phi'(0)] + W\log\log\frac{1}{t} + W + o(1),$$

i.e. $$A(h) < -G + W(1-\gamma) + W\log[-\phi'(0)] + o(1). \tag{18}$$

Since G is arbitrarily large in (18), the conclusion (7) of Theorem a, follows at once from (18).

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FIVE FAMILIES OF RULED SURFACES THROUGH A LINE OF A RECTILINEAR CONGRUENCE, III

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Five families of ruled surfaces through a line of a rectilinear congruence obtained by the repeated application of the operation of forming the Jacobian from Sannia's quadratic forms $f \equiv G_{\alpha\beta}\, du^\alpha\, du^\beta$ and $\phi \equiv \xi_{\alpha\beta}\, du^\alpha\, du^\beta$ were studied in detail by Ogura [7, 8], Hayashi [2], Ram Behari [9] and Mishra [4]; and those from Kummer's quadratic forms $f \equiv G_{\alpha\beta}\, du^\alpha\, du^\beta$ and $\theta \equiv \mu_{\alpha\beta}\, du^\alpha\, du^\beta$ by Mishra [5]. It is the purpose of the present paper to study the ruled surfaces obtained by the repeated application of the operation of forming the Jacobian from the quadratic forms θ and ϕ.

1. Let the co-ordinates of a point on the surface of reference of a rectilinear congruence be given by x^i $(i = 1, 2, 3)$, and the direction cosines of the ray 'l' of the congruence be given by λ^i $(i = 1, 2, 3)$. Suppose

$$G_{\alpha\beta} = \lambda^i_{,\alpha} \cdot \lambda^i_{,\beta}\ ;\ \mu_{\alpha\beta} = \lambda^i_{,\alpha} \cdot x^i_{,\beta} \text{ and } \xi_{\alpha\beta} = E^{\gamma\delta}\, G_{\alpha\gamma}\, \mu_{\delta\beta}, \tag{1.1}$$

where comma (,) followed by an index denotes covariant differentiation with regard to u^α and $E^{\alpha\beta}$ is a tensor conjugate to $E_{\alpha\beta} = (\lambda^i\, \lambda^i_{,\alpha}\, \lambda^i_{,\beta})$.

We have, then, the following quadratic forms :

$$f \equiv G_{\alpha\beta}\, du^\alpha\, du^\beta, \tag{1.2}$$

$$\theta \equiv \mu_{\alpha\beta}\, du^\alpha\, du^\beta, \tag{1.3}$$

$$\phi \equiv \xi_{\alpha\beta}\, du^\alpha\, du^\beta. \tag{1.4}$$

The quadratic forms (1.2) and (1.3) are Kummer's quadratic forms [3], and the quadratic forms (1.2) and (1.4) are Sannia's quadratic forms (Bianchi [1]).

We shall now study the repeated application of the operation of forming the Jacobian from the quadratic forms (1.3) and (1.4).

(a) $\theta \equiv \mu_{\alpha\beta}\, du^\alpha\, du^\beta = 0$ denotes the equation of surfaces whose lines of striction lie on the surface of reference.

(b) $\phi \equiv \xi_{\alpha\beta}\, du^\alpha\, du^\beta = 0$ denotes the equation of developable surfaces of the congruence.

(c) $J(\theta, \phi) = 0$ is the same as

$$\begin{vmatrix} (\mu_{\alpha 1} + \mu_{1\alpha})\, du^\alpha & (\mu_{\alpha 2} + \mu_{2\alpha})\, du^\alpha \\ (\xi_{\beta 1} + \xi_{1\beta})\, du^\beta & (\xi_{\beta 2} + \xi_{2\beta})\, du^\beta \end{vmatrix} = 0,$$

or

$$\epsilon^{\gamma\delta}(\mu_{\gamma\alpha} + \mu_{\alpha\gamma})\,(\xi_{\beta\delta} + \xi_{\delta\beta})\, du^\alpha\, du^\beta = 0, \tag{1.5}$$

where

$$\epsilon^{11} = \epsilon^{22} = 0 \text{ and } \epsilon^{12} = -\epsilon^{21} = 1.$$

It is to be noted that the surfaces (1.5) are the surfaces for which the ratio between the distance to the central point from the surface of reference and the parameter of distribution is extremum [6]. These surfaces coincide with characteristic surfaces if the surface of reference is the middle surface, and with the surfaces $J\{\theta, J(f, \theta)\} = 0$, in case the congruence is normal.

(d) Let us now consider the surfaces

$$J(\theta, J(\theta, \phi)) = 0.$$

These surfaces can be written as

$$\begin{vmatrix} \epsilon^{\gamma\delta}[(\mu_{\gamma 1} + \mu_{1\gamma})(\xi_{\beta\delta} + \xi_{\delta\beta}) + (\mu_{\gamma\beta} + \mu_{\beta\gamma})(\xi_{1\delta} + \xi_{\delta 1})]\, du^\beta & \epsilon^{\gamma\delta}[(\mu_{\gamma 2} + \mu_{2\gamma})(\xi_{\beta\delta} + \xi_{\delta\beta}) + (\mu_{\gamma\beta} + \mu_{\beta\gamma})(\xi_{2\delta} + \xi_{\delta 2})]\, du^\beta \\ (\mu_{\alpha 1} + \mu_{1\alpha})\, du^\alpha & (\mu_{\alpha 2} + \mu_{2\alpha})\, du^\alpha \end{vmatrix} = 0.$$

To investigate the properties of these surfaces, we have from the theory of quadratic forms

$$J(\theta, J(\theta, \phi)) \equiv \Delta\theta.\, \phi - \tfrac{1}{2}F(\theta, \phi).\, \theta = 0, \tag{1.6}$$

where

$$\Delta\theta = \tfrac{1}{8}\, \epsilon^{\alpha\beta}\, \epsilon^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\mu_{\beta\delta} + \mu_{\delta\beta}), \tag{1.7}$$

and

$$F(\theta, \phi) = \tfrac{1}{4}\, \epsilon^{\alpha\beta}\, \epsilon^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta}). \tag{1.8}$$

With the help of (1.7) and (1.8), the equation (1.6) assumes the form

$$\frac{\phi}{\theta} = \frac{\epsilon^{\alpha\beta}\, \epsilon^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta})}{\epsilon^{\alpha\beta}\, \epsilon^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\mu_{\beta\delta} + \mu_{\delta\beta})}, \tag{1.9}$$

or

$$\frac{\xi_{\alpha\beta}\, du^\alpha\, du^\beta}{\mu_{\alpha\beta}\, du^\alpha\, du^\beta} = \frac{E^{\alpha\beta}\, E^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta})}{E^{\alpha\beta}\, E^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\mu_{\beta\delta} + \mu_{\delta\beta})}, \tag{1.10}$$

which, with the help of (1.1), assumes the form

$$\frac{\xi_{\alpha\beta}\, du^\alpha\, du^\beta}{\mu_{\alpha\beta}\, du^\alpha\, du^\beta} = \frac{2\xi_{\gamma\delta}\, G^{\gamma\delta}\, \mu_{\alpha\beta}\, G^{\alpha\beta}}{E^{\alpha\beta}\, E^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\mu_{\beta\delta} + \mu_{\delta\beta})}. \tag{1.11}$$

If d_1 and d_2 are the principal parameters and t_1 and t_2 are the distances to the central point from the surface of reference, then [10, pp. 52, 57]

$$\left.\begin{aligned} d_1 + d_2 &= \xi_{\alpha\beta}\, G^{\alpha\beta}\ ;\ t_1 + t_2 = -\,\mu_{\alpha\beta}\, G^{\alpha\beta}; \\ k \equiv d_1\, d_2 &= \tfrac{1}{8}\, E^{\alpha\beta}\, E^{\gamma\delta}\, (\xi_{\alpha\gamma} + \xi_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta}); \\ t_1\, t_2 &= \tfrac{1}{8}\, E^{\alpha\beta}\, E^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\mu_{\beta\delta} + \mu_{\delta\beta}); \end{aligned}\right\} \tag{1.12}$$

and then equation (1.11) becomes

$$-\frac{\xi_{\alpha\beta}\, du^\alpha\, du^\beta}{\mu_{\alpha\beta}\, du^\alpha\, du^\beta} = \frac{1}{4}\cdot \frac{(t_1 + t_2)\, (d_1 + d_2)}{t_1\, t_2}. \tag{1.13}$$

The left hand member of this equation is the ratio between the parameter of distribution of the surface and the distance to the central point from the surface of reference.

From equation (1.13) *we see that the surfaces* $J(\theta, J(\theta, \phi)) = 0$ *coincide with developable surfaces, if either the congruence is normal or the surface of reference is the middle surface.*

(e) The surfaces $J(\phi, J(\theta, \phi)) = 0$ can be written as

$$\begin{vmatrix} \epsilon^{\gamma\delta}\{(\mu_{\alpha1} + \mu_{1\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta}) + & \epsilon^{\gamma\delta}\{(\mu_{\alpha2} + \mu_{2\alpha})\, (\xi_{\delta\delta} + \xi_{\delta\beta}) + \\ + (\mu_{\gamma\beta} + \mu_{\beta\gamma})\, (\xi_{1\delta} + \xi_{\delta1})\}du^\beta & + (\mu_{\gamma\beta} + \mu_{\beta\gamma})\, (\xi_{2\delta} + \xi_{\delta2})\}du^\beta \\ (\xi_{\alpha1} + \xi_{1\alpha})du^\alpha & (\xi_{\alpha2} + \xi_{2\alpha})du^\alpha \end{vmatrix} = 0.$$

Also, from the theory of quadratic forms, these surfaces can be written as

$$\Delta\, \phi.\, \theta - \tfrac{1}{2}\, F(\theta, \phi).\, \phi = 0, \tag{1.14}$$

where

$$\Delta\, \phi = \tfrac{1}{8}\, \epsilon^{\alpha\beta}\, \epsilon^{\gamma\delta}\, (\xi_{\alpha\gamma} + \xi_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta}).$$

Hence the equation (1.14) assumes the form

$$\frac{\theta}{\phi} = \frac{E^{\alpha\beta}\, E^{\gamma\delta}\, (\mu_{\alpha\gamma} + \mu_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta})}{E^{\alpha\beta}\, E^{\gamma\delta}\, (\xi_{\alpha\gamma} + \xi_{\gamma\alpha})\, (\xi_{\beta\delta} + \xi_{\delta\beta})}, \tag{1.15}$$

which, with the help of (1.12), becomes

$$-\frac{\theta}{\phi} = \frac{1}{4}\frac{(t_1 + t_2)\,(d_1 + d_2)}{k}\,. \tag{1.16}$$

From the equation (1.16) we see that *the surfaces*

$$J(\phi, J(\theta, \phi)\,) = 0$$

coincide with surfaces whose lines of striction lie on the surface of reference, provided the congruence is normal or the surface of reference is the middle surface.

2. Let us now summarise these results together with the results of the previous papers on the same subject [4], [5].

(a) With the quadratic forms f and ϕ, we have

(i) surfaces whose spherical representations are minimal lines $[f = 0]$;

(ii) developable surfaces $[\phi = 0]$;

(iii) surfaces of distribution $[J(f, \phi) = 0]$;

(iv) principal surfaces $[J\,(f, J(f, \phi)) = 0$ or $2d = h$, where d is the parameter of distribution and h is the mean parameter$]$;

(v) characteristic surfaces $[J(\phi, J(f, \phi)) = 0$ or $2\,k = hd]$.

(b) With the quadratic forms f and θ we have

(i) surfaces whose spherical representations are minimal lines $[f = 0]$;

(ii) surfaces whose lines of striction lie on the surface of reference $[\theta = 0]$;

(iii) principal surfaces $[J(f, \theta) = 0]$;

(iv) surfaces of distribution $[J(f, J(f, \theta)) = 0$ or $t = \frac{1}{2}(t_1 + t_2)]$;

(v) surfaces for which $t = \dfrac{2t_1 t_2}{t_1 + t_2}$ $[J(\theta, J(f, \theta)) = 0]$.

(c) With quadratic forms θ and ϕ, we have

(i) surfaces whose lines of striction lie on the surface of reference $[\theta = 0]$;

(ii) developable surfaces $[\phi = 0]$;

(iii) surfaces for which the ratio between the parameter of distribution and the distance to the central point from the surface of reference is extremum $[J(\theta, \phi) = 0]$;

(iv) surfaces for which the above ratio is given by

$$p = \frac{(t_1 + t_2)(d_1 + d_2)}{4\, t_1 t_2} \qquad [J(\theta, J(\theta, \phi)) = 0];$$

(v) surfaces for which $p = \dfrac{4k}{(t_1 + t_2)(d_1 + d_2)}$ $[J(\phi, J(\theta, \phi)) = 0]$.

An examination of the above results indicates that given three quadratic forms f, θ and ϕ we get *ten* distinct surfaces by the repeated formation of the Jacobian from these forms taken two at a time. They are

$f = 0$; $\phi = 0$; $J(f, \phi) = 0$ or $J(f, J(f, \theta)) = 0$; $J(f, J(f, \phi)) = 0$ or $J(f, \theta) = 0$; $J(\phi, J(f, \phi)) = 0$;
$\theta = 0$; $J(\theta, J(f, \theta)) = 0$;
$J(\theta, \phi) = 0$; $J(\theta, J(\theta, \phi)) = 0$; $J(\phi, J(\theta, \phi)) = 0$.

When the surface of reference is the middle surface,

$J(f, \phi) = 0$, $J(\theta, J(f, \theta)) = 0$, $J(\theta, \phi) = 0$, $J(\theta, J(\theta, \phi)) = 0$, $J(\phi, J(\theta, \phi)) = 0$ coincide with $\theta = 0$, $f = 0$, $J(\phi, J(f, \phi)) = 0$, $\phi = 0$, $\theta = 0$ respectively, that is, the surfaces reduce to *five* only.

In case the congruence is normal,

$J(f, J(f, \phi)) = 0$, $J(\phi, J(f, \phi)) = 0$, $J(\theta, \phi) = 0$, $J(\theta, J(\theta, \phi)) = 0$, $J(\phi, J(\theta, \phi)) = 0$ coincide with $\phi = 0$, $f = 0$, $J(\theta, J(f, \theta)) = 0$, $\phi = 0$, $\theta = 0$ respectively.

The surfaces in this case again reduce to *five* only.

3. We shall now investigate some properties of these surfaces. We know that the central planes for the mean ruled surfaces bisect the angles between the principal planes [10, p. 58].

Let the developable surfaces be represented by parametric lines, then we have

$$\xi_{\alpha\alpha} = 0. \tag{3.1}$$

The equation of the surfaces of distribution is

$$J(f, \phi) = 0,$$

or

$$\begin{vmatrix} (\xi_{1\alpha} + \xi_{\alpha 1})\, du^{\alpha} & (\xi_{2\alpha} + \xi_{\alpha 2})\, du^{\alpha} \\ G_{1\beta}\, du^{\beta} & G_{2\beta}\, du^{\beta} \end{vmatrix} = 0. \tag{3.2}$$

Applying the condition (3.1) in (3.2), we get

$$\begin{vmatrix} G_{1\alpha}\, du^{\alpha} & G_{2\alpha}\, du^{\alpha} \\ du^2 & du^1 \end{vmatrix} = 0,$$

or

$$G_{11}\,(du^1)^2 - G_{22}\,(du^2)^2 = 0, \tag{3.3}$$

which gives the result :

The central planes for the spherical representations of the surfaces of distribution through a line of a rectilinear congruence bisect the angles between the central planes for the spherical representations of developable surfaces.

The equation of characteristic surfaces is

$$J(\phi, J(f, \phi)) = 0. \tag{3.4}$$

The necessary and sufficient condition for the surface of reference to be the middle surface is

$$\mu_{\alpha\beta} G^{\alpha\beta} = 0 \quad \text{or} \quad \epsilon^{\alpha\beta} \xi_{\alpha\beta} = 0. \tag{3.5}$$

In that case, the equation (3.4) assumes the form

$$\begin{vmatrix} (\mu_{\alpha 1} + \mu_{1\alpha})\, du^{\alpha} & (\mu_{\alpha 2} + \mu_{2\alpha})\, du^{\alpha} \\ \xi_{1\beta} du^{\beta} & \xi_{2\beta} du^{\beta} \end{vmatrix} = 0. \qquad (3.6)$$

This surface is parametric if

$$\mu_{\alpha\beta} + \mu_{\beta\alpha} = \xi_{\alpha\beta} + \xi_{\beta\alpha} = 0, \qquad \alpha \neq \beta. \qquad (3.7)$$

The equation of the surfaces of distribution will then take the form

$$\mu_{\alpha\beta} du^{\alpha} du^{\beta} = 0, \qquad \alpha = \beta. \qquad (3.8)$$

With the help of (3.7), the equation (3.5) assumes the form

$$\mu_{\alpha\beta} G^{\alpha\beta} = 0, \qquad \alpha = \beta. \qquad (3.9)$$

By virtue of (3.9), the equation (3.8) becomes

$$G_{11}(du^1)^2 - G_{22}(du^2)^2 = 0.$$

Hence we also get the result :

The central planes for the spherical representations of the surfaces of distribution through a line of a rectilinear congruence also bisect angles between the central planes for the spherical representations of characteristic surfaces, provided the surface of reference is the middle surface.

The surfaces $J(\theta, J(f, \theta)) = 0$ are given by

$$\begin{vmatrix} \{\epsilon^{\gamma\delta}[G_{1\gamma}(\mu_{\beta\delta} + \mu_{\delta\beta}) + & \{\epsilon^{\gamma\delta}[G_{2\gamma}(\mu_{\beta\delta} + \mu_{\delta\beta}) + \\ + G_{\beta\gamma}(\mu_{1\delta} + \mu_{\delta 1})]\, du^{\beta}\} & + G_{\beta\gamma}(\mu_{2\delta} + \mu_{\delta 2})]\, du^{\beta}\} \\ (\mu_{1\alpha} + \mu_{\alpha 1})\, du^{\alpha} & (\mu_{2\alpha} + \mu_{\alpha 2})\, du^{\alpha} \end{vmatrix} = 0.$$

These surfaces are parametric if

$$\mu_{\alpha\beta} + \mu_{\beta\alpha} = 0, \qquad \alpha \neq \beta, \qquad (3.10)$$

and

$$\epsilon^{\gamma\delta}\,[G_{\alpha\gamma}(\mu_{\beta\delta} + \mu_{\delta\beta}) + G_{\beta\gamma}(\mu_{\alpha\delta} + \mu_{\delta\alpha})] = 0, \qquad \alpha \neq \beta. \qquad (3.11)$$

The condition (3.11) is the same as

$$G_{11}\mu_{22} - G_{22}\mu_{11} = 0. \qquad (3.12)$$

The principal surfaces are given by

$$\begin{vmatrix} G_{1\alpha}du^{\alpha} & G_{2\alpha}du^{\alpha} \\ (\mu_{1\beta}+\mu_{\beta 1})\,du^{\beta} & (\mu_{2\beta}+\mu_{\beta 2})\,du^{\beta} \end{vmatrix} = 0. \quad (3.13)$$

In consequence of (3.10) and (3.12), the equation (3.13) assumes the form

$$\mu_{11}(du^1)^2 - \mu_{22}(du^2)^2 = 0,$$

or with the help of (3.12), we get

$$G_{11}(du^1)^2 - G_{22}(du^2)^2 = 0.$$

Hence *the central planes for the spherical representations of principal surfaces bisect the angles between the central planes for the spherical representations of the surfaces* $J\{\theta, J(f, \theta)\} = 0$.

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A CHARACTERIZATION OF MINIMALLY BICOMPACT SPACES

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In this paper a necessary and sufficient condition for a space to be minimally bicompact is established. Also the question of the existence of a minimally bicompact topology weaker than a given bicompact topology is considered, and it is shown that the set P_t of points at which a (bicompact) topology t is minimally bicompact can be increased (by weakening it) by any scattered set.

The following characterization of minimally bicompact spaces is related to a result of Alexandroff [1], namely that in a bicompact space every point has a convergence character, which is equivalent to: 'for every closed set X for which p is an accumulation point, there is a subset Y of X which flows towards p'; for, from Theorem 1 below it follows that in a minimally bicompact space, for every accumulation point p of a closed set X, there is a subset Y of X which flows only towards p. (For definition of 'flowing' see [1, p. 268]).

THEOREM 1. *The space R is minimally bicompact if and only if for every limit point p, of a closed set A, there exists a subset Z of A not containing p but of which p is the sole complete limit point* [2].

Let R be minimally bicompact, and suppose that p is a limit point of the closed set A, but for Z in A not containing p, p is not the sole complete limit point of Z. Then Z must have a complete limit point z in A (A being bicompact) and $z \neq p$. Let us introduce as additional open sets the sets of the form

$V \cup p$, where V is any open set disjoint from A. Then every infinite set Z consists of two parts: $Z_1 \supset A$ and Z_2 disjoint from A. One of these parts has the same power as Z. If it is Z_1 then it has a complete limit point $z_1 \neq p$. If it is Z_2 it has a complete limit point z outside A. In either case the point found is still a complete limit point in the expanded topology so that the latter is bicompact, a contradiction, since in the latter, p is not a limit point of A.

For the converse, suppose p is a limit point of $S \subset R$, and thus of its closure A. Let t' be a topology on R retaining all the open sets of R. Suppose that p is not a limit point of S in t', as must be true for some p and S if t' is not the topology of R. Then there is a t'-open set V containing p, meeting neither A nor S. Now $A-p$ contains a set Z of which p is the sole complete limit point, and which consequently has no limit point in t'. Hence t' is not bicompact, and R itself is proved minimally bicompact.

It is evident that the above condition applies to minimal bicompactness at a point—R being defined to be minimally bicompact at the point p if no topology weaker than the given topology at p is bicompact [3, p. 26], i.e. *the bicompact space R is minimally bicompact at the point p if and only if every closed set A of which p is a limit point has a subset Z not containing p, but of which p is the sole complete limit point.*

For any bicompact topology t let P_t denote the set of all points of R at which (R, t) is minimally bicompact. (P_t may be null). Then we have

THEOREM 2. *If (R, t_0) is bicompact, and A is any scattered set (i.e. a set which does not contain any dense-in-itself subset), there exists a bicompact topology t' weaker than or equal to t_0, differing from t_0 only at points of A, such that $P_{t'} \supset A$, (or equivalently $P_{t'} \supset A \cup P_t$).*

Let $A_0 = A - D(A)$; $A_1 = (A - A_0) - D(A - A_0)$; ... , for any ordinal β, $A_\beta = (A - \sum_{\gamma<\beta} A_\gamma) - D(A_\gamma - \sum_{\gamma<\beta} A_\gamma)$; ..., (where $D(Z)$ for any set Z denotes the derivative of Z). Then it is evident that each A_β is an isolated set. Also since A is scattered, $A = \sum_{\gamma<\delta} A_\gamma$, for some ordinal δ.

Let $T = \{t_\gamma\}_{\gamma<\alpha}$ be an ascending chain (i.e. such that each element t_γ is weaker than all the preceding elements of T) in the class C of all bicompact topologies which are weaker than or equal to t_0 and differ from t_0 only at points of A. Let t_α be the lattice product of $[t_\gamma]_{\nu<\alpha}$. Then t_α is weaker than t_0 and differs from t_0 only at points of A, and hence $t_\alpha \in C$ if (R, t_α) is bicompact. Suppose if possible that (R, t_α) is not bicompact. Then there exists a set X which has no complete limit point in (R, t_α). Let $B_\gamma(X)$ denote the set of complete limit points of X in (R, t_γ). Since each t_γ is bicompact $B_\gamma(X) \neq 0$, and since each t_γ is weaker than the preceding elements of T, $\{B_\gamma(X)\}_{\gamma<\alpha}$ is a decreasing series. $\prod_{\gamma<\alpha} B_\gamma(X) = B_\alpha(X)$, for since t_α is weaker than each t_γ, $B_\alpha(X) \subset \prod_{\gamma<\alpha} B_\gamma(X)$, and since t_α is the lattice product of $[t_\gamma]_{\gamma<\alpha}$ the reverse relation also holds, i.e.

$$\prod_{\gamma<\alpha} B_\nu(X) = B_\alpha(X). \tag{1}$$

It also follows from (1) that since $B_\alpha(X) = 0$,

$$\prod_{\gamma<\alpha} B_\gamma(X) = 0. \tag{2}$$

Since $B_\alpha(X) = 0$, $B_0(X)$ (and hence $B_\gamma(X)$ for each $\gamma<\alpha$) $\subset A$, for if $p \in B_0(X) - A$, since $t_\alpha = t_0$ at p, $p \in B_\alpha(X)$ also, i.e. $B_\alpha(X) \neq 0$, contrary to hypothesis. Hence

$$B_0(X) \subset A. \tag{3}$$

Consider the special case where A is an isolated set; i.e. $A = A_0$. $B_0(X)$ is closed (being a local function) and is isolated

(being contained in A by result (3)). Hence $B_0(X)$ is discrete. Since (R, t_0) is bicompact, $B_0(X)$ is finite. Let $B_0(X) = \sum_{r<n^*} p_r$. Let γ_r denote the lowest value of γ for which $B_\gamma(X)$ does not contain p_r. γ_r exists for each $r \leqslant n$, since otherwise $\prod_{\gamma<\alpha} B_\gamma(X) \neq 0$, contrary to result (1) above. The largest ordinal γ' of the set $[\gamma_r]_{r\leqslant n} < \alpha$ since α is a limiting ordinal. Also since $[B_\gamma(X)]_{\gamma<\alpha}$ is a decreasing series, $B_{\gamma'}(X) = 0$, contrary to the hypothesis that $t_{\gamma'}$ is bicompact for $\gamma' < \alpha$. Hence the hypothesis that (R, t_α) is not bicompact is wrong, i.e. the chain T has a maximal element in C. Hence there exists a topology t' weaker than or equal to t_0, differing from t_0 only at points of A, such that (R, t') is minimally bicompact at all points of A or equivalently, such that $P_{t'} \supset P_t \cup A$, if A is an isolated set. (4)

The general result follows from (4) by induction, for in (4) consider $A = A_0$. There exists a bicompact topology t_1, weaker than t_0, differing from t_0, only at points of A_0, such that $P_{t_1} \supset P_{t_0} \cup A_0$. Suppose that this result holds for A_γ for all $\gamma < \mu$, i.e. for all $\gamma < \mu$ there exists a bicompact topology t_γ, weaker than or equal to each $t_{\gamma'}$ for $\gamma' < \gamma$, differing from $t_{\gamma'}$ only at points of $\sum_{\gamma'\leqslant\beta<\gamma} A_\beta$, such that $P_{t_\gamma} \supset P_{t_{\gamma'}} \cup \sum_{\gamma'\leqslant\beta<\gamma} A_\beta$. Then this result holds for $\gamma = \mu$ also, for consider the case where μ is a non-limiting ordinal $= \lambda + 1$. Since A_λ is an isolated set, by result (4) above, there exist a bicompact topology $t_{\lambda+1}$ weaker than (or $=$) t_λ, differing from t_λ only at points of A_λ, such that $P_{t_{\lambda+1}} \supset P_{t_\lambda} \cup A_\lambda$, or equivalently, weaker than or $= t_\gamma$, and differing from t_γ only at points of $\sum_{\gamma\leqslant\beta<\lambda+1} A_\beta$ for all $\gamma < \lambda + 1$, and such that $P_{t_{\lambda+1}} \supset P_{t_\gamma}$ $\sum_{\gamma\leqslant\beta<\lambda+1} A_\beta$ (since t_λ satisfies the conditions of the hypothesis). Hence the result holds for $\mu = \lambda + 1$ also. Consider the case where μ is a limiting ordinal. Let t_μ be the lattice product of $[t_\gamma]_{<\mu\gamma}$. Then for each $\gamma < \mu$, t_μ is weaker than t_γ (or $= t_\gamma$), and

differs from t_γ only at points of $\sum_{\gamma \leqslant \beta < \mu} A_\beta$, and is minimally bicompact at all points of

$$P_{t_\gamma} \cup \sum_{\gamma \leqslant \beta < \mu} A_\beta = P_{t_0} \cup \sum_{\beta < \mu} A_\beta$$

if it is bicompact. If t_μ is not bicompact, there exists a set X for which $B_\mu(X)$, the set of complete limit points of X in (R, t_μ) is null. Since each t_γ $(\gamma < \mu)$ is bicompact, $B_\mu(X)$ is non-null for all $\gamma < \mu$. But by (1) $B_\mu(X) = \prod_{\gamma < \mu} B_\gamma(X)$. Each $B_\gamma(X) \supset B_0(X) \cup \sum_{\beta \geqslant \gamma} A_\beta$ since $t_\gamma = t_0$ at all points of $\sum_{\beta < \gamma} A_\beta$ and hence of $\sum_{\beta \geqslant \gamma} A_\beta$. Therefore since $B_\mu(X) = 0$,

$$\prod_{\gamma < \mu} [B_0(X) \cap \sum_{\beta \geqslant \gamma} A_\beta] = 0.$$

But each set $B_0(X) \cap \sum_{\beta \geqslant \gamma} A_\beta$ is closed in (R, t_0) ($\sum_{\beta \geqslant \gamma} A_\beta$ being closed since no accumulation point of $\sum_{\beta \geqslant \gamma} A_\beta$ belongs to $\sum_{\beta < \gamma} A_\beta$). Therefore since (R, t_0) is bicompact, $B_0(X) \cap \sum_{\beta \geqslant \gamma'} A_\beta = 0$, for some $\gamma' < \mu$, i.e. $B_0(X) \subset \sum_{\beta < \gamma'} A_\beta$ by result (3). t_μ differs from $t_{\gamma'}$ only at points of $\sum_{\gamma' \leqslant \beta < \mu} A_\beta$, and hence

$$B_\mu(X) \cap \sum_{\beta < \gamma'} A_\beta = B_{\gamma'}(X) \cap \sum_{\beta < \gamma'} A_\beta.$$

The right side $= B_{\gamma'}(X)$, since $B_{\gamma'}(X) \subset B_0(X) \subset \sum_{\beta < \gamma'} A_\beta$, and the left side $= 0$ since $B_\mu(X) = 0$. Hence $B_{\gamma'}(X) = 0$, contrary to the hypothesis that $(R, t_{\gamma'})$ is bicompact. Hence the result holds for $\gamma = \mu$ if it holds for all $\gamma < \mu$, in all cases. But the result has been proved above for $\gamma = 1$. Hence by induction it holds for all γ, and hence for $\gamma = \delta$ also, i.e. there exists a bicompact topology t' weaker than (or $=$) t_0, differing from t_0 only at points of A, such that $P_{t'} \supset P_{t_0} \cup A$.

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THE GENERAL UNIQUENESS THEOREM AND SUCCESSIVE APPROXIMATIONS

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1. Introduction. Picard's classical theorem on successive approximations to the solution of a differential equation $y' = f(x, y)$ assumes that the function f satisfies Lipschitz's condition. In this case not only do the successive approximations converge, but the solution is unique. Leaving aside the question of convergence of successive approximations, the question of uniqueness was studied by Osgood, Montel, Nagumo, Phillippov and others. All the above cases are, however, only particular cases of a general theorem given by Kamke [1]. This theorem runs as follows :

Let

(1) $\omega(x, z)$ *be continuous and non-negative in the region defined by* $0 < x \leqslant a$, $z \geqslant 0$;

(2) $\chi(x) \equiv 0$ *be the only integral of the differential equation* $z' = \omega(x, z)$ *in the interval* $0 < x < \alpha \leqslant a$, *which has the property*

$$\lim_{x \to 0} \chi(x) = \lim_{x \to 0} \frac{\chi(x)}{x} = 0 ;$$

(3) $f(x, y)$ *be continuous in the region* $0 \leqslant x \leqslant a$; $|y| \leqslant b$, *and*

$$|f(x, y_1) - f(x, y_2)| \leqslant \omega(x, |y_1 - y_2|).$$

Then the solution of the differential equation $y' = f(x, y)$ *through* $(0, 0)$ *is unique.*

But the mere uniqueness of the solution does not ensure the convergence of successive approximations, as was shown by Müller [2]. Nor is the converse true, as can be seen by the familiar example $y' = \sqrt{(xy)}$, which has an infinite number of

solutions through $(0, 0)$, while the successive approximations converge to $y \equiv 0$. The question now naturally arises: under what conditions do the successive approximations converge? It was shown by Aurel Wintner [3] that if the differential equation satisfies Osgood's condition for uniqueness, the successive approximations converge on a sufficiently small interval. Van Kampen [4] proved the same for Nagumo's condition. Dieudonné [5] considered the above theorem of Kamke, and proved the convergence of successive approximations with the following additional conditions:

(i) $\omega(x, z)$ is monotonic increasing in z;

(ii) the only solution $\chi(x)$ of the equation $z' = \omega(x, z)$ such that $\lim_{x\to 0} \chi(x) = \liminf_{x\to 0} \dfrac{\chi(x)}{x} = 0$ is $\chi(x) \equiv 0$.

Condition (ii) is more restrictive than condition (2) of Kamke's theorem, and Dieudonné himself, in a foot-note, suggests the consideration of the theorem with only one additional condition, namely that $\omega(x, z)$ is monotonic increasing in z. The object of this paper is to prove the convergence of the successive approximations with only this additional condition.

2. Statement of the theorem. *Let*

(1) $\omega(x, z)$ *be continuous and non-negative, in the region defined by* $0 < x \leqslant a$; $z \geqslant 0$, *and monotonic increasing in* z *for any fixed value of* x;

(2) $\chi(x) \equiv 0$ *be the only intergal of the differential equation* $z' = \omega(x, z)$ *in the interval* $0 < x \leqslant a$ *which has the property* $\lim_{x\to 0} \chi(x) = \lim_{x\to 0} \dfrac{\chi(x)}{x} = 0$;

(3) $f(x, y)$ *be continuous in the region* $0 \leqslant x \leqslant a$; $|y| \leqslant b$ *and* $|f(x, y_1) - f(x, y_2)| \leqslant \omega(x, |y_1 - y_2|)$.

Then the successive approximations to the solution of the differential equation $y' = f(x, y)$ *through* $(0, 0)$ *given by*

$$y_{n+1}(x) = \int_0^x f(x, y_n)\, dx$$

converge on a sufficiently small interval $0 \leqslant x \leqslant c$ to its unique solution.

3. Proof of the theorem. Following Aurel Wintner, we put

$$\mu(x) = \limsup_{n \to \infty} |y_{n+1}(x) - y_n(x)| . \tag{3.1}$$

To prove the theorem we have only to show that (see [3] or [5])

$$\mu(x) \equiv 0. \tag{3.2}$$

We require the following properties of $\mu(x)$ to deduce (3.2).

I. $\mu(x)$ is continuous and $\mu(0) = 0$;

II. If x_0 is any point in the interval $0 < x \leqslant a$, for $0 < x \leqslant x_0$, we have

$$\mu(x) \geqslant \mu(x_0) - \int_x^{x_0} \omega\{s, \mu(s)\} ds;$$

III. $\lim_{x \to 0} \frac{\mu(x)}{x} = 0;$

IV. If $\psi(x)$ is a solution of the differential equation $z' = \omega(x, z)$ through the point $(x_0, \mu(x_0))$, then in the interval $0 < x \leqslant x_0 \leqslant a$, we have $\mu(x) \geqslant \psi(x)$.

Property I can be easily verified.

PROOF OF PROPERTY II. For $x > x_0$ consider the value of $\mu(x) - \mu(x_0)$.

$$\begin{aligned}
\mu(x) - \mu(x_0) &= \limsup_{n \to \infty} |y_{n+1}(x) - y_n(x)| - \limsup_{n \to \infty} |y_{n+1}(x_0) - \\
&\qquad\qquad - y_n(x_0)| \\
&\leqslant \limsup_{n \to \infty} (|y_{n+1}(x) - y_n(x)| - |y_{n+1}(x_0) - y_n(x_0)|) \\
&\leqslant \limsup_{n \to \infty} |y_{n+1}(x) - y_{n+1}(x_0) - y_n(x) + y_n(x_0)| \\
&\leqslant \limsup_{n \to \infty} \left| \int_{x_0}^{x} f\{s, y_n(s)\} \, ds - \int_{x_0}^{x} f\{s, y_{n-1}(s)\} \, ds \right| \\
&\leqslant \limsup_{n \to \infty} \int_{x_0}^{x} \omega(s, |y_n - y_{n-1}|) \, ds \\
&\leqslant \int_{x_0}^{x} \omega\{s, \mu(s)\} \, ds,
\end{aligned}$$

by Fatou's theorem, and on account of the monotonic increasing character of $\omega(x, z)$ in z, as was shown by Aurel Wintner [3]. Therefore

$$\mu(x) \leqslant \mu(x_0) + \int_{x_0}^{x} \omega\{s, \mu(s)\}\, ds.$$

To get the corresponding result for the interval to the left of x_0, we interchange x and x_0, and after transposing, obtain

$$\mu(x) \geqslant \mu(x_0) - \int_{x}^{x_0} \omega\{s, \mu(s)\}\, ds, \quad \text{for } 0 < x \leqslant x_0.$$

PROOF OF PROPERTY III.

$$\frac{\mu(x)}{x} = \limsup_{n\to\infty} \left|\frac{y_{n+1}-y_n}{x}\right| = \limsup_{n\to\infty}\left|\frac{1}{x}\int_0^x \{f(s,y_n)-f(s,y_{n-1})\}\, ds\right|$$
$$= \limsup_{n\to\infty} |f\{\xi, y_n(\xi)\} - f\{\xi, y_{n-1}(\xi)\}|,$$

where ξ depends on n, and $0 < \xi(n) \leqslant x$ for all values of n. Therefore, in order to determine the behaviour of $\frac{\mu(x)}{x}$ near $x = 0$, it is enough to determine the behaviour of

$$\limsup_{n\to\infty} |f\{x, y_n(x)\} - f\{x, y_{n-1}(x)\}|$$

near $x = 0$.

We shall show that every one of the terms of this sequence can be made less than an arbitrarily given small positive quantity ϵ, by choosing x less than a certain positive quantity independent of n. Then the limit superior of this sequence also can be made less than ϵ, which proves that $\lim_{x\to 0} \frac{\mu(x)}{x} = 0$.

To demonstrate this, we observe that

$$|f(x, y_n) - f(x, y_{n-1})| \leqslant |f(x, y_n) - f(0, 0)| + |f(x, y_{n-1}) - f(0, 0)|.$$

Since $f(x, z)$ is continuous at the origin, given $\epsilon/2$ it is possible to find two positive quantities σ and ρ such that

$$|f(x, y_n) - f(0, 0)| < \epsilon/2, \text{ for } |x| < \sigma \text{ and } |y_n| < \rho.$$

Further, since $y_n(0) = 0$, and $\{y_n(x)\}$ are equicontinuous, it is possible to find a τ such that

$$|y_n(x)| = |y_n(x) - y_n(0)| < \rho, \text{ for } x < \tau,$$

where τ is independent of n. Choose $x < a = \min(\sigma, \tau)$. Then for $x < a$, we have

$$|f\{x, y_n(x)\} - f\{x, y_{n-1}(x)\}| < \epsilon$$

and since $\xi_{(n)} \leqslant x$ for all value of n,

$$\lim_{x \to 0} \frac{\mu(x)}{x} = 0.$$

PROOF OF PROPERTY IV. Suppose that at any point ξ of the interval $0 < \xi < x_0$, we have $\mu(\xi) < \psi(\xi)$. On account of the continuity of both the functions, and since they have the same value at x_0, there is a greatest open interval containing ξ in which this inequality holds. Suppose that this open interval is $0 < x_1 < x < x_2 < x_0$. At x_1 and x_2 the two functions have identical values, i.e. $\mu(x_1) = \psi(x_1)$ and $\mu(x_2) = \psi(x_2)$. Since $\omega(x, z)$ is monotonic increasing in z, we have in this open interval

$$\omega\{x, \mu(x)\} < \omega\{x, \psi(x)\};$$

therefore

$$\mu(x_2) - \int_x^{x_2} \omega\{s, \mu(s)\}\, ds > \mu(x_2) - \int_x^{x_2} \omega\{s, \psi(s)\}\, ds$$
$$= \psi(x),$$

since $\psi(x)$ is a solution of $z' = \omega(x, z)$ through $\{x_2, \mu(x_2)\}$, in the interval to the left of x_2. Since Property II is true for every point x_0 in the interval,

$$\mu(x) \geqslant \mu(x_2) - \int_x^{x_2} \omega\{s, \mu(s)\}\, ds$$
$$> \psi(x).$$

This is a contradiction to our supposition. Hence there is no point at which $\mu(\xi) < \psi(\xi)$ holds. In other words, throughout the interval $0 < x \leqslant x_0$, we have $\mu(x) \geqslant \psi(x)$.

We are now in a position to prove (3.2). Suppose that $\mu(x)$ has a non-zero value at x_0. Consider a solution of the differential equation $z' = \omega(x, z)$ through the point $\{x_0, \mu(x_0)\}$ in the interval $0 < x \leqslant x_0$. Let us denote this by $\psi(x)$.

By IV we have in this interval, $0 < \psi(x) \leqslant \mu(x)$. Since $\mu(x)$ exists at all points of this interval including zero, and is continuous, $\psi(x)$ is finite at all these points, and since $\mu(0) = 0$, $\lim_{x\to 0} \psi(x) = 0$. The passage to the limit is justified by a general theorem. See [1, p. 74, Theorem 1]. Further

$$0 < \frac{\psi(x)}{x} \leqslant \frac{\mu(x)}{x}, \quad \text{for} \quad 0 < x \leqslant x_0,$$

and since $\lim_{x\to 0} \frac{\mu(x)}{x} = 0$, we get at once $\lim_{x\to 0} \frac{\psi(x)}{x} = 0$.

Therefore $\psi(x)$ is a solution of the differential equation which satisfies the condition

$$\lim_{x\to 0} \psi(x) = \lim_{x\to 0} \frac{\psi(x)}{x} = 0.$$

But $\psi(x)$ has a non-zero value at x_0, which contradicts the hypothesis. Hence $\mu(x) \equiv 0$, and the successive approximations converge. This completes the proof of the theorem.

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UNIQUENESS AND THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS

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Although the uniqueness of the solution of an initial value problem of ordinary differential equations does not imply the convergence of the successive approximations (cf. the example due to Müller, [4]), it is known that many of the sufficient conditions for uniqueness, with a certain requirement of monotonicity, do imply the convergence of these approximations to the solution. For example, van Kampen [5] showed that a generalization of Nagumo's condition for uniqueness is sufficient for the convergence of the successive approximations ; Wintner [6] proved that the Osgood criterion for uniqueness (with the monotone requirement) is also sufficient. Recently La Salle [3] combined the features of the Osgood and Nagumo hypotheses to obtain a sufficient condition for the convergence of the successive approximations which includes as special cases the Wintner and van Kampen results, although he was evidently unaware of the latter result.

There is a very general well-known uniqueness theorem [2, p. 139] which includes the Osgood and Nagumo theorems, and their obvious generalizations, and it is the purpose of this note to indicate that the hypotheses of this theorem, with the additional assumption of monotonicity, also imply the convergence of the successive approximations (cf. Theorem I below). Actually we prove a slightly more general result (Theorem II below) which is in the spirit of the Carathéodory existence theorem [1, p. 672]. The proof will depend on an adaptation of the proofs of Wintner's theorem and the general uniqueness theorem quoted. The result includes, as special cases, those

cited above. After this paper was completed (early in 1950), it was brought to our attention that Dieudonné [*Bull. Sci. Math.*, 69 (1945), 62-72] had considered this problem, and our result was then laid aside. Only recently did we discover through the work of Mr. Viswanatham [this *Journal*, pp. 69-74] that our theorems not only included Dieudonné's result, but, in fact, take care of the objection that Dieudonné himself raised (p. 64, footnote 3) to his theorem.

Let y and $f(x, y)$ be vectors with real components $(y_1, \ldots, y_n)$, $\{f_1(x, y), \ldots, f_n(x, y)\}$, and for vectors y, let $|y| = \sum_{i=1}^{n} |y_i|$. We consider the system with initial condition

$$y' = f(x, y),\ y(0) = 0, \tag{1}$$

where $f(x, y)$ is continuous on the region

$$R : |x| < a,\ |y| < b, \tag{2}$$

and

$$|f(x, y)| \leqslant M,\ (x, y) \text{ in } R. \tag{3}$$

If $y_0(x)$ is any initial vector for which $y_0(0) = 0$, the successive approximations $y_m(x)$ for (1) are defined by

$$y_m(x) = \int_0^x f\{t, y_{m-1}(t)\}\, dt. \tag{4}$$

THEOREM I. *Let $\phi(x, r) \geqslant 0$ be continuous, non-decreasing in r for fixed x, on*

$$0 < x < a,\ r \geqslant 0; \tag{5}$$

and suppose for each α, $0 < \alpha \leqslant a$, $\rho(x) \equiv 0$ is the only differentiable function on $0 \leqslant x < \alpha$ which satisfies

$$\rho'(x) = \phi\{x, \rho(x)\},\ 0 < x < \alpha, \tag{6}$$

and

$$\rho(0) = \rho'(0) = 0. \tag{7}$$

Consider the differential equation with initial condition (1) *where $f(x, y)$ is continuous, bounded, on R* (cf. (2) *and* (3)), *and, satisfies there (for $x \neq 0$)*

$$|f(x, \tilde{y}) - f(x, y)| \leqslant \phi(|x|, |\tilde{y} - y|). \tag{8}$$

Then, for every continuous initial vector $y_0(x)$ *such that* $y_0(0) = 0$, $|y_0(x)| < b$ *when* $|x| \leqslant \beta < \min(a, b/M)$, *the successive approximations* (4) *converge (uniformly) on* $|x| \leqslant \beta$ *to the solution of* (1), *as* $m \to \infty$.

THEOREM II. *Let* $\phi(x, r) \geqslant 0$ *be measurable in* x *for fixed* $r > 0$, *continuous, non-decreasing, in* r *for fixed* x, $0 < x < a$; *and for every bounded subset of* (5) *let there exist a* $\psi(x)$ *such that*

$$\phi(x, r) \leqslant \psi(x), \tag{9}$$

where $\psi(x)$ *is integrable on* $\gamma < x < a$, *for every* $\gamma > 0$. *Suppose for each* α, $0 < \alpha < a$, $\rho(x) \equiv 0$ *is the only absolutely continuous function on* $0 \leqslant x < \alpha$ *which satisfies*

$$\rho'(x) = \phi\{x, \rho(x)\} \tag{10}$$

almost everywhere on $0 < x < \alpha$, *and such that* $\rho'(0)$ *exists, and*

$$\rho(0) = \rho'(0) = 0. \tag{11}$$

Then, if, with this $\phi(x, r)$, $f(x, y)$ *satisfies the same conditions as in Theorem* I, *the conclusion of Theorem* I *is valid.*

REMARKS. The hypotheses on $f(x, y)$ and $\phi(x, r)$ in Theorem I, *except* the monotone restriction in r on the latter function are precisely those of the general uniqueness theorem referred to above [2, p. 139]. Therefore it makes sense to speak of *the* solution of (1) in this case. Actually uniqueness follows also from the hypotheses of Theorem II. The proof of this fact is essentially contained in the proof of Theorem II, and therefore will not be given explicitly. We prove only Theorem II, since it contains Theorem I.

PROOF OF THEOREM II. Choose, once for all, any $y_0(x)$ satisfying the hypothesis of the theorem. It follows from (3) and (4), that the sequence $\{y_m(x)\}$ is uniformly bounded and equicontinuous on $|x| \leqslant \beta$. Hence there exists a subsequence $\{y_{m_k}(x)\}$ which is uniformly convergent to a function $y(x)$ as $k \to \infty$. The subsequence

$$y_{m_k+1}(x) = \int_0^x f\{t, y_{m_k}(t)\}\, dt$$

is uniformly convergent on $|x| \leqslant \beta$ to the function

$$y^*(x) = \int_0^x f\{t, y(t)\}dt.$$

We prove below that $y_{m+1}(x) - y_m(x) \to 0$, $m \to \infty$, on $|x| \leqslant \beta$. Assuming this, it follows that $y_{m_k+1}(x) - y_{m_k}(x) \to 0$, $k \to \infty$, on $|x| \leqslant \beta$, and this implies $y^*(x) = y(x)$, that is, $y(x)$ is a solution of (1) on $|x| \leqslant \beta$. Because of uniqueness, every subsequence of $\{y_m(x)\}$ which is convergent will tend to the same solution, and this proves that $\{y_m(x)\}$ is convergent on $|x| \leqslant \beta$ to the unique solution of (1).

In order to prove $y_{m+1}(x) - y_m(x) \to 0$, $m \to \infty$, on $|x| \leqslant \beta$, define

$$w_m(x) = y_{m+1}(x) - y_m(x)$$

and

$$v(x) = \limsup_{m \to \infty} |w_m(x)|.$$

Now $v(0) = 0$, and $v(x)$ is continuous on $|x| \leqslant \beta$ for it is the upper limit of an equicontinuous, uniformly bounded sequence of functions. To prove $w_m(x) \to 0$, $m \to \infty$, on $|x| \leqslant \beta$, is equivalent to showing $v(x) \equiv 0$ on $|x| \leqslant \beta$. We do this for $0 \leqslant x \leqslant \beta$; the proof for $-\beta \leqslant x \leqslant 0$ is similar.

For any function $g(x)$, let $\Delta g(x) = g(x + \Delta x) - g(x)$ where $\Delta x > 0$. Then from (4) it follows that for $0 < x < \beta$,

$$|\Delta w_{m+1}(x)| \leqslant \int_x^{x+\Delta x} |f\{t, y_{m+1}(t)\} - f\{t, y_m(t)\}| \, dt,$$

and because of (8)

$$|\Delta w_{m+1}(x)| \leqslant \int_x^{x+\Delta x} \phi\{(t, |w_m(t)|\} \, dt. \tag{12}$$

Given any $\delta > 0$, there exists an integer $N(\delta)$, independent of t and m, such that

$$|w_m(t)| \leqslant v(t) + \delta, \quad m > N(\delta), \tag{13}$$

for all t in $0 < x \leqslant t \leqslant x + \Delta x \leqslant \beta$. This follows from the fact that $v(t)$ is uniformly continuous, and $\{w_m(t)\}$ is equicontinuous on $x \leqslant t \leqslant x + \Delta x$.

Since $\phi(x, r)$ is non-decreasing in r, it follows from (13) that

$$\phi\{t, |w_m(t)|\} \leqslant \phi\{t, v(t) + \delta\}, \quad m > N(\delta), \tag{14}$$

and consequently, using (12),

$$|\Delta w_{m+1}(x)| \leqslant \int_x^{x+\Delta x} \phi\{t, v(t) + \delta\}\, dt, \quad m > N(\delta). \tag{15}$$

From the definition of $v(x)$ it is easy to see that

$$|\Delta v(x)| \leqslant \limsup_{m \to \infty} |\Delta w_m(x)|, \tag{16}$$

and this with (15) shows that

$$|\Delta v(x)| \leqslant \int_x^{x+\Delta x} \phi\{t, v(t) + \delta\} dt. \tag{17}$$

On account of the continuity of $\phi(x, r)$ in r,

$$\phi\{t, v(t) + \delta\} \to \phi\{t, v(t)\}, \quad \text{as } \delta \to 0;$$

and from (9), and the dominated convergence theorem of Lebesgue, it follows that

$$\lim_{\delta \to 0} \int_x^{x+\Delta x} \phi\{t, v(t) + \delta\} dt = \int_x^{x+\Delta x} \phi\{t, v(t)\} dt.$$

The last relation, along with (17), yields

$$|\Delta v(x)| \leqslant \int_x^{x+\Delta x} \phi\{t, v(t)\} dt. \tag{18}$$

The inequality (18) implies that $v(x)$ is absolutely continuous over any interval $[x, x + \Delta x]$, $x > 0$, and consequently $v'(x)$ exists almost everywhere on $[x, x + \Delta x]$; from (18), this derivative satisfies

$$|v'(x)| \leqslant \phi\{x, v(x)\}, \quad x > 0, \tag{19}$$

almost everywhere. Suppose for some ξ, $0 < \xi \leqslant \beta$, that $v(\xi) \neq 0$. It is not too difficult to see that the initial value problem $r' = \phi(x, r)$, $r(\xi) = v(\xi) > 0$, possesses an absolutely continuous solution $r(x)$ which exists on some interval to the left of $x = \xi$, say $\xi - \sigma \leqslant x \leqslant \xi$, $\sigma > 0$ (cf. [1], p. 672). Now, on this interval

$$r(x) \leqslant v(x); \tag{20}$$

for if this were not the case, there would exist a first point to the left of $x = \xi$, say $x = \zeta$, where $r(\zeta) = v(\zeta)$, and $r(x) > v(x)$ for $x < \zeta$, and sufficiently near $x = \zeta$. For small enough $h > 0$, it follows from (19) that

$$v(\zeta) - v(\zeta - h) \leqslant \int_{\zeta-h}^{\zeta} \phi\{t, v(t)\}\, dt, \tag{21}$$

and from the definition of $r(x)$,

$$v(\zeta) - r(\zeta - h) = \int_{\zeta-h}^{\zeta} \phi\{t, r(t)\}\, dt. \tag{22}$$

Since $\phi(x, r)$ is non-decreasing in r, $\phi\{t, v(t)\} \leqslant \phi\{t, r(t)\}$ on $\zeta - h \leqslant t \leqslant \zeta$, and this together with (21) and (22) implies that $r(\zeta - h) \leqslant v(\zeta - h)$, which is a contradiction. This establishes the inequality (20).

Now $r(x) > 0$ on $0 < x < \xi$ as far as it exists ; for if $r(\bar{\xi}) = 0$ for some $0 < \bar{\xi} < \xi$, the function

$$\bar{\rho}(x) = \begin{cases} 0, \ 0 \leqslant x \leqslant \bar{\xi}, \\ r(x), \ \bar{\xi} < x \leqslant \xi, \end{cases}$$

would be a solution of (10) and (11) on $0 \leqslant x < \xi$, distinct from $\rho(x) \equiv 0$, and this contradicts the hypothesis of the theorem. Therefore

$$0 < r(x) \leqslant v(x), \tag{23}$$

as far to the left of $x = \xi$ as $r(x)$ exists. But by (23) and (9) $r(x)$ must exist on the *whole* interval $0 < x \leqslant \xi$. Hence $\lim_{x \to 0} r(x)$ exists, and by (23) this limit is 0. Define $r(0) = 0$.

From (23) we have

$$0 < r(x)/x \leqslant v(x)/x, \qquad 0 < x \leqslant \xi. \tag{24}$$

It will now be shown that $v(x)/x \to 0$, as $x \to 0$. For this purpose consider, for $x > 0$,

$$\frac{w_m(x)}{x} = \frac{y_{m+1}(x) - y_m(x)}{x} = \frac{1}{x}\int_0^x [f\{t, y_m(t)\} - f\{(t, y_{m-1}(t)\}\, dt. \tag{25}$$

Since $f(x, y)$ is continuous on R, and from (3) and (4), $|y_m(t)| \leqslant Mt$, it follows from (25) that given any $\epsilon > 0$ there exists an $\eta(\epsilon) > 0$ such that $|w_m(x)| < \epsilon\, x$, whenever $0 \leqslant x \leqslant \eta(\epsilon)$; and hence $\limsup_{m \to \infty} |w_m(x)| = v(x) \leqslant \epsilon\, x$, if $0 \leqslant x \leqslant \eta(\epsilon)$, or $v(x)/x \to 0$, $x \to 0$.

From (24), it now follows that $r(x)/x \to 0$, $x \to 0$, or, since $r(0) = 0$, $r'(0) = 0$. This contradicts the hypotheses of the theorem, for $r(x)$ is an absolutely continuous solution of $r' = \phi(x, r)$, $0 < x < \xi$, which is not identically zero on $0 \leqslant x < \xi$. Therefore $v(\xi) \neq 0$ for any ξ, $0 < \xi \leqslant \beta$, and this proves the theorem.

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THE DERIVATIVES OF CERTAIN FUNCTIONS

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It was shown by V. G. Iyer [3] that *if* $f(x)$ *is analytic on the closed interval* $[a, b]$, *and if for some point* $x_0 \in [a, b]$ *the sequence of derivatives* $f^{(n)}(x_0)$ *has a limit*: $f^{(n)}(x_0) \to l$, *then* $f^{(n)}(x) \to l\, e^{x-x_0}$ *uniformly for* x *on* $[a, b]$. Boas and Chandrasekharan [1], [2] reach the same conclusion with weaker assumptions on $f(x)$. We propose to consider some extensions of this result of Iyer.

One may restate the conclusion of the Iyer theorem as follows : *On* $[a, b]$, *and in fact on an arbitrary bounded region* R *of the complex plane*, $f(x)$ *has the representation*

$$f(x) = Ae^x + \phi(x), \tag{1}$$

where A *is a constant* ($A = l\, e^{-x_0}$) *and* $\phi(x)$ *is an entire function such that* $\phi^{(n)}(x) \to 0$ *uniformly on* R. If $f(x)$ is analytic, and the number of limit points of the sequence $\{f^{(n)}(x_0)\}$ is finite and greater than one, it cannot be concluded that $f(x)$ has a representation analogous to that of (1). However, if we have information about the limit points of $\{f^{(n)}(x)\}$ at *two points*, then there is the following parallel with the Iyer theorem :

THEOREM 1. *Let the entire function*[1] $f(z)$ *have the property that at two points* z_1, z_2 *the sequences* $\{f^{(n)}(z_1)\}$, $\{f^{(n)}(z_2)\}$ *have only a finite number of limit points (each limit point being finite). Then* $f(z)$ *has the representation*

$$f(z) = E(z) + \phi(z), \tag{2}$$

where for some integer m, $E(z)$ *has the form*

$$E(z) = \sum_{k=0}^{m-1} A_k \exp\{\omega^k z\} \tag{3}$$

(1) If $f(z)$ is given as analytic in a region or on a curve (e g on a segment of the real axis), and if at some point z_0 of the region or curve the sequence $\{f^{(n)}(z_0)\}$ is bounded, then $f(z)$ can be continued analytically to be an entire function. It is therefore no restriction to accept $f(z)$ as entire to begin with.

with[2] $\omega = \exp \{2\pi i/m\}$, *and* $\phi(z)$ *is an entire function such that* $\phi^{(n)}(z) \to 0$ *uniformly on every bounded set.*

PROOF. There is no loss of generality in supposing that $z_1 = 0$. We shall also use the letter a instead of z_2. If we set

$$f(z) = \sum_{n=0}^{\infty} x_n. z^n/n! \quad , \tag{4}$$

then $x_n = f^{(n)}(0)$; and if we let $y_n = f^{(n)}(a)$, we see from (4) that

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \cdot x_{n+k} = y_n, \ (n = 0, 1, 2, \ldots). \tag{5}$$

If we show that

$$x_n = x_n^* + \theta_n \quad , \tag{6}$$

where $\{x_n^*\}$ is a periodic sequence and $\theta_n \to 0$, then

$$f(z) = \sum_{n=0}^{\infty} x_n^* \frac{z^n}{n!} + \sum_{n=0}^{\infty} \theta_n \frac{z^n}{n!}.$$

The first series on the right side will represent a function $E(z)$ for some[3] m, and the second series an entire function $\phi(z)$ such that $\phi^{(n)}(0) \to 0$ and therefore such that $\phi^{(n)}(z) \to 0$ uniformly on an arbitrary bounded set[4] in the complex plane. We therefore need only establish (6) where x_n^*, θ_n have the specified properties. Relation (6) is a consequence of the following more general result that we shall now establish :

THEOREM 2. *Let*[5] $\{a_n\}$, $n = 0, 1, \ldots$ *be a sequence satisfying the conditions* (i) $a_n \neq 0$, $n \geqslant N$; (ii) $\limsup_{n\to\infty} |a^{n+1}/a_n| = \rho < 1$. *Let* $\{x_n\}$, $\{y_n\}$ *satisfy the equations*

$$\sum_{k=0}^{\infty} a_k x_{n+k} = y_n \qquad (n = 0, 1, \ldots), \tag{7}$$

(2) we set $e^u \equiv \exp\{u\}$.

(3) For if $F(z) = \sum_0^{\infty} x_n^* \, z^n/n!$ and m is the order of periodicity of $\{x_n^*\}$, then $F^{(m)}(z) = F(z)$.

(4) It is easy to see that $\phi^{(n)}(0) \to 0$ implies $\phi^{(n)}(z) \to 0$ uniformly on every bounded set R.

(5) Throughout this paper all quantities not otherwise given by statement or context as real, are assumed to lie in the field of complex numbers.

and suppose $\{x_n\}$ *is bounded and has only q limit points:* $s_1, \ldots, s_q$, *and that* $\{y_n\}$ *has only a finite number of limit points. If*

$$\mu\rho < \sigma(1-\rho), \tag{8}$$

where[6]

$$\mu = \max|s_i - s_j|, \quad \sigma = \min|s_i - s_j| \qquad (i \neq j),$$

then

$$x_n = x_n^* + \theta_n', \quad y_n = y_n^* + \theta_n'' \tag{9}$$

where $\theta_n' \to 0$, $\theta_n'' \to 0$ *and* $\{x_n^*\}$, $\{y_n^*\}$ *are periodic sequences having a period in common.*

Remark. If $\rho = 0$, which is to say that $a_{n+1}/a_n \to 0$, then (8) automatically holds.

Proof of Theorem 2. By hypothesis there is an M for which

$$|x_n| \leqslant M \qquad (n = 0, 1, \ldots),$$

so also

$$|s_j| \leqslant M \qquad (j = 1, \ldots, q).$$

If non-overlapping circles of arbitrarily small radius are drawn with centres at $s_1, \ldots, s_q$, then all but a finite number of $\{x_n\}$ will lie in these circles, so there is an $N_1 \geqslant N$ such that for each $n \geqslant N_1$ the number x_n is closest to some one of the quantities $s_1, \ldots, s_q$, say $s_{h(n)}$; and likewise, this N_1 can be chosen so that for $n \geqslant N_1$ each y_n is closest to some one of the limit points[7] $t_1, \ldots, t_r$ of $\{y_n\}$, say $t_{k(n)}$. Then for $n \geqslant N_1$,

$$x_n = X_n + \psi_n', \quad y_n = Y_n + \psi_n'', \tag{10}$$

where $X_n = s_{h(n)}$, $Y_n = t_{k(n)}$, and where $\psi_n' \to 0$, $\psi_n'' \to 0$.

From (7) and (10) we obtain

$$\sum_{k=0}^{\infty} a_k X_{n+k} = Y_n + \omega_n, \tag{11}$$

(6) The definitions of σ, μ presuppose that $q > 1$. The case $q = 1$ is trivial, for if $q = 1$ then for all n, $x_n = s_1 + \theta_n'$, $(\theta_n' \to 0)$; therefore from (7), $y_n = t_1 + \theta_n''$ where $t_1 = s_1\left(\sum_{k=0}^{\infty} a_k\right)$ and $\theta_n'' = \Sigma\, a_k\, \theta'_{n+k} \to 0$. We may therefore assume $q > 1$

(7) The hypothesis of Theorem 2 does not state that the limit points of $\{y_n\}$ are all finite, but this is a consequence of relations (7), in view of condition (ii) and the fact that $\{x_n\}$ is bounded.

where

$$\omega_n = \psi_n'' - \sum_{k=0}^{\infty} a_k \psi'_{n+k} .$$

It is clear from condition (ii) that $\sum_{k=0}^{\infty} |a_k| < \infty$, so $\omega_n \to 0$. Choose ϵ so that $0 < \epsilon < 1 - \rho$, otherwise arbitrary. Then from (ii) it is seen that there is a $K_1 \geq N_1$ such that if $n \geq K_1$, then

$$\sum_{k=1}^{\infty} |a_{n+k}/a_n| < \frac{\rho + \epsilon}{1 - (\rho + \epsilon)} .$$

Let δ be chosen, as it can, to satisfy

$$\frac{\rho}{1-\rho} < \delta < \frac{\sigma}{\mu} .$$

Then ϵ can be chosen small enough so that for $n \geq K_1$ (K_1 corresponding to this choice of ϵ),

$$\sum_{k=1}^{\infty} |a_{n+k}/a_n| < \delta. \tag{12}$$

Again, there is a $K_2 \geq N_1$ such that if $m \geq K_2$, then

$$\sum_{k=m}^{\infty} |a_k| < \sigma_2/(2\mu), \tag{13}$$

where[8]

$$\sigma_2 = \min |t_i - t_j| \qquad (i \neq j).$$

Let $K = \max \{K_1, K_2\}$, so that $n \geq K$, $m \geq K$ imply both (12) and (13). Having determined K, we can then find $K_3 \geq K$, such that if $n \geq K_3$ then

$$|\omega_n| < \min \left\{ \frac{\sigma_2 a}{4}, \frac{(\sigma - \mu\delta)a}{2} \right\} \tag{14}$$

where

$$a = \min \{1, |a_K|\}.$$

Define

$$P_n = a_0 X_n + a_1 X_{n+1} + \dots + a_{K-1} X_{n+K-1}.$$

Since each X_n is one of the numbers $s_1, \dots, s_q$, it follows that there are not more than q^K different expressions P_n, so there certainly exist two different values $n = u, w$, both exceeding

(8) The purpose of (13) and (14) is to establish that $Y_u = Y_w$ in (15). Now if $\{y_n\}$ has only one limit point, then σ_2 is not defined, but in this case we have Y_n=constant, so $Y_u = Y_w$ anyway.

K_3, such that term for term we have $P_u = P_w$. Taking $n = u, w$ in (11) and subtracting, we obtain

$$\sum_{k=0}^{q} a_{k+K}\{X_{u+k+K} - X_{w+k+K}\} = \{Y_u - Y_w\} + \{\omega_u - \omega_w\}. \quad (15)$$

From (13) we see that the left side of (15) is in magnitude less than $\mu\left(\frac{\sigma_2}{2\mu}\right) = \sigma_2/2$; and from (14), $|\omega_u - \omega_w| < \sigma_2/2$. Hence

$$|Y_u - Y_w| < \sigma_2. \quad (16)$$

Since each Y_n is one of the numbers $t_1, \ldots, t_r$, (16) contradicts the definition of σ_2 unless $Y_u = Y_w$. Returning then to (15), we have,

$$\{X_{u+K} - X_{w+K}\} = \frac{1}{a_K}\{\omega_u - \omega_w\} - \sum_{k=1}^{\infty} \frac{a_{k+K}}{a_K}\{X_{u+k+K} - X_{w+k+K}\}. \quad (17)$$

The first term on the right is in magnitude less than $(\sigma - \mu\delta)$ by (14), and the sum is, by (12), less in magnitude than $\mu\delta$. Thus

$$|X_{u+K} - X_{w+K}| < \sigma. \quad (18)$$

As argued in (16), this requires that $X_{u+K} = X_{w+K}$.

Now take $n = u + 1, w + 1$ in (11). Since we now have $P_{u+1} = P_{w+1}$, we can apply the same reasoning, obtaining (15) with u, w replaced by $u + 1, w + 1$, and therefore $Y_{u+1} = Y_{w+1}$, and finally $X_{u+1+K} = X_{w+1+K}$. And so on, the general case leading to the equality,

$$X_{u+K+m} = X_{w+K+m} \qquad (m = 0, 1, 2, \ldots). \quad (19)$$

Thus, at least beginning with $n = u$ the sequence $\{X_n\}$ is periodic with a (not necessarily primitive) period $w - u$; and the very same thing is true of $\{Y_n\}$.

Define $\{x_n^*\}$, $\{y_n^*\}$, $n = 0, 1, \ldots$ as periodic sequences such that

$$x_n^* = X_n, \quad y_n^* = Y_n \qquad (n \geqq u),$$

and define θ_n', θ_n'' for $n = 0, 1, \ldots$ so that (9) holds. Then

$$\theta_n' = \psi_n', \quad \theta_n'' = \psi_n'' \qquad (n \geqq u),$$

so

$$\theta_n' \to 0, \quad \theta_n'' \to 0.$$

This establishes the theorem.

By specializing the sequence $\{a_n\}$ one can obtain various consequences of Theorem 2 in addition to Theorem 1. The following is an example of this :

COROLLARY. *Let $0 < |c_1| \leqslant \ldots \leqslant |c_{s-1}| < |c_s|$ and let $\lambda_j \neq 0$, $j = 1, \ldots, s$. If the entire function $f(z)$ has the property that $\{f^{(n)}(0)\}$ is bounded and that the two sequences $\{f^{(n)}(0)\}$, $\{\lambda_1 f^{(n)}(c_1) + \ldots + \lambda_s f^{(n)}(c_s)\}$ have only a finite number of limit points, then $f(z)$ has a representation of the form* (3) *of Theorem* 1 *where $E(z)$ (for some m) and $\phi(z)$ have the properties stated in that theorem.*

PROOF. Setting $f(z) = \sum_{n=0}^{\infty} x_n \frac{z^n}{n!}$ and $a_k = \frac{1}{k!} \sum_{j=1}^{s} \lambda_j c_j^k$, we have

$$\sum_{k=0}^{\infty} a_k x_{n+k} = y_n,$$

where $y_n = \sum_{j=1}^{s} \lambda_j f^{(n)}(c_j)$. All the hypotheses of Theorem 2 are seen to hold, and therefore also the conclusion, and this gives the corollary.

Let us return to Theorem 1, where it was shown that $f(z)$ has a representation (2). One may inquire as to what information we have regarding the "order", as we shall term the number m, of the exponential sum (3). The proof of Theorem 2 shows that a possible choice of m is the number $w - u$, but we know little about the size of this number, and anyway it may be much bigger[9] than is necessary. The following result gives a partial solution to the problem of determining a value for m :

(9) If an exponential sum, of the form $E(z)$ of (3), is of " order " m, it is also of " order " hm where h is an arbitrary positive integer. Consequently, if we have the (or better, an) " order " of $E(z)$, we have no assurance that this " order " is the smallest possible.

THEOREM 3. *Let the function $f(z)$ have the property that for the two points z_1, z_2 the sequences $\{f^{(n)}(z_1)\}$, $\{f^{(n)}(z_2)\}$ have only a finite number of limit points, namely $s_1, \dots, s_q$; $t_1, \dots, t_r$ respectively*[10]. *Let*

$$\sigma_1 = \min|s_i - s_j|,\ \sigma_2 = \min|t_i - t_j|,\ \mu_1 = \max|s_i - s_j|,\ \mu_2 = \max|t_i - t_j| \qquad (i \neq j);$$

$$R_j = \mu_j/\sigma_j\ ,\ j = 1, 2.$$

Let $\delta_j = \delta_{R_j}$ be the unique positive zero of the function

$$\Delta_j(\delta) = \delta - R_j\,(e^\delta - 1 - \delta), \qquad (j = 1, 2). \quad (20)$$

If

$$|z_1 - z_2| < \min\left\{\delta_1, \log\left(1 + \frac{\sigma_2}{\mu_1}\right)\right\}, \quad (21)$$

then

$$f(z) = E(z) + \phi(z), \quad (22)$$

where $E(z)$ is an exponential sum of "order" q:

$$E(z) = \sum_{k=0}^{q-1} A_k \exp\{\omega^k z\} \quad (23)$$

with $\omega = \exp\{2\pi i/q\}$, and $\phi(z)$ is an entire function such that $\phi^{(n)}(z) \to 0$ uniformly in every bounded region.

REMARK. Since the "order" in (23) is q, which is the number of limit-points at the one point z_1, it is perhaps not surprising that some such limitation as (21) is necessary. That *some* restriction on $|z_1 - z_2|$ is indispensable is seen from the following example:

$$f(z) = \sin z + \cos z = \left(\frac{1-i}{2}\right)e^{iz} + \left(\frac{1+i}{2}\right)e^{-iz},$$

with $z_1 = 0$, $z_2 = \pi$. The limit points of $\{f^{(n)}(z_1)\}$ are ± 1, so $q = 2$. If the conclusion of Theorem 3 were to hold with no condition on $|z_1 - z_2|$, then we would have

$$f(z) = Ae^z + Be^{-z} + \phi(z)$$

where $\phi^{(n)}(z) \to 0$. But this is manifestly impossible.

(10) We may suppose that $q > 1$, $r > 1$ since otherwise the Iyer theorem applies.

PROOF OF THEOREM 3. $f(z)$ is clearly an entire function. Expanding $f(z)$ about $z = z_1$, we have

$$f^{(k)}(z) = \sum_{n=0}^{\infty} f^{(k+n)}(z_1)(z - z_1)^n / n! \qquad (k = 0, 1, \ldots). \qquad (24)$$

Define $s_{h(n)}$, $t_{k(n)}$ as in the proof of Theorem 2, so that

$$f^{(n)}(z_1) = s_{h(n)} + \psi_n', f^{(n)}(z_2) = t_{k(n)} + \psi_n''$$

with $\psi_n' \to 0$, $\psi_n'' \to 0$; then set

$$x^*_n = s_{h(n)}, \; y_n^* = t_{k(n)}.$$

From (24) we see that given $\epsilon > 0$, there exists a $K_0 = K_0(\epsilon)$ such that if $k > K_0$ then

$$\left|f^{(k)}(z) - \sum_{n=0}^{\infty} x^*_{n+k}(z - z_1)^n / n!\right| < \epsilon. \exp\{|z - z_1|\}; \qquad (25)$$

and since

$$|\psi_k''| = |f^{(k)}(z_2) - y^*_k| < \epsilon$$

for k sufficiently great, there is a $K_1 > K_0$ such that if $k > K_1$ then

$$\left|y^*_k - \sum_{n=0}^{\infty} x^*_{n+k}(z_2 - z_1)^n / n!\right| < \epsilon[1 + \exp\{|z_2 - z_1|\}]. \qquad (26)$$

Denote the sequence of integers n for which $x_n^* = s_j$ by $\{n(j)\}$, $j = 1, \ldots, q$. We distinguish two cases :

I. Suppose a value $j = m$ exists such that when k runs through the integers of the sequence $\{n(m)\}$ then infinitely many of the integers $k + 1$ lie in $\{n(a)\}$ and infinitely many in $\{n(b)\}$, for some $a \neq b$ (with $a = m$ or $b = m$ permitted). There then exists an infinite sequence $\{k_n\}$, $n = 1, 2, \ldots$ such that $k_n > K_1$, $k_n \in \{n(m)\}$ and $(k_{2p} + 1) \in \{n(a)\}$, $(k_{2p+1} + 1) \in \{n(b)\}$.

If in (26) we take $k = k_{2p}$, k_{2p+1} and subtract, and recall that $x^*_{k_{2p}} = x^*_{k_{2p+1}}$ (since k_{2p} and k_{2p+1} both belong to $\{n(m)\}$), we get

$$\left|\{y^*_{k_{2p}} - y^*_{k_{2p+1}}\} - \sum_{n=1}^{\infty}\{x^*_{n+k_{2p}} - x^*_{n+k_{2p+1}}\}\frac{(z_2 - z_1)^n}{n!}\right| <$$

$$< 2\epsilon[1 + \exp\{|z_2 - z_1|\}]. \quad (27)$$

Hence if $y^*_{k_{2p}} - y^*_{k_{2p+1}} \neq 0$, then

$$\sigma_2 \leqslant |y^*_{k_{2p}} - y^*_{k_{2p+1}}| < 2\epsilon[1 + \exp\{|z_2 - z_1|\}] + \\ + \mu_1[\exp\{|z_2 - z_1|\} - 1].$$

Since σ_2 is independent of ϵ, we have

$$\sigma_2 \leqslant \mu_1[\exp\{|z_2 - z_1|\} - 1];$$

so

$$|z_2 - z_1| \geqslant \log(1 + \sigma_2/\mu_1).$$

But this contradicts (21). Hence

$$y^*_{k_{2p}} = y^*_{k_{2p+1}}.$$

Relation (27) therefore reduces to

$$\left|(z_2 - z_1)\{x^*_{1+k_{2p}} - x^*_{1+k_{2p+1}}\} + \sum_{n=2}^{\infty}\{x^*_{n+k_{2p}} - x^*_{n+k_{2p+1}}\}\frac{(z_2 - z_1)^n}{n!}\right| \\ < 2\epsilon[1 + \exp\{|z_2 - z_1|\}]. \quad (28)$$

Now $x_{1+k_{2p}}{}^* \neq x_{1+k_{2p+1}}{}^*$, since the left side is a member of $\{n(a)\}$ and the right side of $\{n(b)\}$. Consequently

$$|z_2 - z_1|\sigma_1 < 2\epsilon[1 + \exp\{|z_2 - z_1|\}] + \mu_1[\exp\{z_2 - z_1|\} - \\ - 1 - |z_2 - z_1|].$$

Again from the independence of the left side with respect to ϵ we get

$$|z_2 - z_1| - R_1[\exp\{|z_2 - z_1|\} - 1 - |z_2 - z_1] \leqslant 0,$$

and this contradicts (21). Thus, Case I is inadmissible.

II. The contrary case, which we now consider, is that in which there are no sequences $\{n(m)\}$, $\{n(a)\}$, $\{n(b)\}$, with $a \neq b$, satisfying the condition of Case I.

Consider $\{n(1)\}$. As k runs through this sequence, $k + 1$ must be contained, except possibly for a finite number of values, in one and the same $\{n(m)\}$. If $m = 1$, then infinitely many pairs k, $k + 1$ lie in $\{n(1)\}$. Consider a particular such k. There exists $a = a_k > 0$ such that $k, k + 1, \ldots, k + a$ all belong to $\{n(1)\}$, but $k + a + 1$ is not in $\{n(1)\}$; for if no such a

exists then $f^{(n)}(z_1)$ has only one limit point,[11] a case handled by the Iyer theorem. Hence $\{n(1)\}$ contains infinitely many pairs $k + a_k - 1$, $k + a_k$ such that $k + a_k + 1$ is not in $\{n(1)\}$. Thus, as k runs through $\{n(1)\}$ there are infinitely many values of $k + 1$ in $\{n(1)\}$ and infinitely many in some $\{n(b)\}$ with $b \neq 1$. But this means that Case I holds, whereas we are assuming Case II. Hence we cannot have $m = 1$. By suitable renumbering we may suppose that $m = 2$. Thus, there is an N_1 such that if $k > N_1$ and $k \in \{n(1)\}$, then $(k + 1) \in \{n(2)\}$.

Now let k run through $\{n(2)\}$. Then $k + 1$ must lie, except possibly for a finite number of values, in some one sequence $\{n(j)\}$. As argued above, we cannot have $j = 2$. But it is also seen that if $q > 2$ then $j \neq 1$. For otherwise there would exist an $N_2 > N_1$ such that if $k > N_2$ and $k \in \{n(1)\}$ then $(k + 1) \in \{n(2)\}$ and $(k + 2) \in \{n(1)\}$; so the sequence of *all* integers greater than N_2 are alternately in $\{n(1)\}$ and $\{n(2)\}$. This means that $q = 2$, so if $q > 2$ we must have $j \neq 1$. By proper renumbering we may suppose that $j = 3$.

Suppose $q > 3$. As k runs through $\{n(3)\}$, $k + 1$ lies in one sequence $\{n(l)\}$ except possibly for a finite number of values of k. As reasoned before, $l \neq 2, 3$. But also $l \neq 1$, else there would exist an $N_3 > N_2$ such that if $k > N_3$ and $k \in \{n(1)\}$, then $(k + 1) \in \{n(2)\}$, $(k + 2) \in \{n(3)\}$, $(k + 3) \in \{n(1)\}$; so no integer greater than N_3 is in $\{n(4)\}$. This means that $q = 3$, contrary to supposition. We may therefore suppose that $l = 4$.

It is clear that this process can be carried out to the point that there is an $N_{q-1} > N_{q-2}$ such that if $k > N_{q-1}$ and $k \in \{n(1)\}$, then $(k + p) \in \{n(p + 1)\}$, $p = 1, 2, \ldots, q - 1$. If we apply the process once more, we find that there exists an $N_q > N_{q-1}$ such that if $k > N_q$ and $k \in \{n(q)\}$ then $(k + 1) \in \{n(p)\}$ for some p.

(11) See the preceding foot note.

Now the possibilities $p = q, q - 1, \ldots, 2$ are ruled out by the reasoning already given. Hence we must have $p = 1$.

Thus, the sequence of all integers exceeding N_q has the property that those integers that are in the same residue class modulo q are in the same sequence $\{n(p)\}$. The integers from 0 to N_q can now be reallocated[12] so that this same result now holds for all integers $n = 0, 1, 2, \ldots$. From the definition of the sequences $\{n(j)\}$, $j = 1, \ldots, q$ it follows that $\{x^*_n\}$ is a periodic sequence with a period q. Since

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_1) \frac{(z - z_1)^n}{n!} = \sum_{n=0}^{\infty} \{x^*_n + \psi_n'\} \frac{(z - z_1)^n}{n!},$$

and $\psi_n' \to 0$, we conclude that $f(z)$ has the representation (22) where $E(z)$ is of "order" q. This completes the proof.

COROLLARY. *If condition* (21) *is replaced by*

$$|z_2 - z_1| < \min \left\{ \delta_2, \log\left(1 + \frac{\sigma_1}{\mu_2}\right) \right\}, \tag{29}$$

then (22) *holds with* $E(z)$ *of "order"* r. *And if* (21) *and* (29) *both hold, then* $E(z)$ *is of "order"* $d = \text{g.c.d.}\{q, r\}$.

PROOF. The first part is obtained by interchanging the roles of z_1, z_2. As for the second part, if $E(z)$ is simultaneously of "order" q and "order" r, then it must have the "order" d = greatest common divisor of q and r. For, the two expressions for $E(z)$ as of "order" q and "order" r must be identical term for term. Hence every exponent $\omega_1^a z$ must be matched by an exponent $\omega_2^b z$, where $\omega_1 = \exp\{2\pi i/q\}$, $\omega_2 = \exp\{2\pi i/r\}$ and $0 \leqslant a < q$, $0 \leqslant b < r$; so $2\pi i a/q = 2\pi i b/r$. Let $q = dq_1$, $r = dr_1$; then $r_1 a = q_1 b$. Now r_1, q_1 are relatively prime, whence it follows that a is a multiple of q_1 and b of r_1. Thus, $\omega_1^a = \omega_2^b = \omega_3^c$ where $\omega_3 = \exp\{2\pi i/d\}$ and $0 \leqslant c < d$. This means that every term in $E(z)$ has an exponent of form $\omega_3^c z$; so $E(z)$ has the "order" d.

(12) This does not violate the hypothesis of the theorem.

An extension of the Iyer theorem in a different direction is given by

THEOREM 4. *Let the following conditions hold:*

(i) *$f(x)$ is of class C^∞ on an interval I of the x-axis.*

(ii) *There is an infinite sequence $\{n_j\}$, $j = 1, 2, \ldots$ such that*

$$\lim_{j\to\infty} f^{(n_j)}(x) = g(x) \tag{30}$$

uniformly[13] *on I.*

(iii) *There is a positive integer k such that the two sequences $\{n_j\}$, $\{n_j - k\}$ have infinitely many integers in common.*

Then $g(x)$ is an exponential sum of "order" k:

$$g(x) = \sum_{r=0}^{k-1} A_r \exp\{\omega^r z\} \tag{31}$$

where $\omega = \exp\{2\pi i/k\}$.

PROOF. Let c be *a* fixed point, and x a variable point, of I. By successive integrations we have

$$f^{(n_j-1)}(x) - f^{(n_j-1)}(c) = \int_c^x f^{(n_j)}(t)\,dt \to \int_c^x g(t)\,dt \equiv G_1(x)$$

$$f^{(n_j-2)}(x) - f^{(n_j-2)}(c) - \frac{(x-c)}{1\,!} f^{(n_j-1)}(c) =$$

$$= \int_c^x [f^{(n_j-1)}(t) - f^{(n_j-1)}(c)]\,dt \to \int_c^x G_1(t)\,dt \equiv G_2(x) \tag{32}$$

$$f^{(n_j-k)}(x) - \sum_{s=1}^{k} f^{(n_j-s)}(c) \frac{(x-c)^{k-s}}{(k-s)!} \to$$

$$\int_c^x G_{k-1}(t)\,dt \equiv G_k(x).$$

The passages of the limit sign across the integral sign are permitted by the uniformity of convergence.

Let $\{m_j\}$ be that infinite subsequence of $\{n_j\}$ for which $\{m_j - k\} \subset \{n_j\}$. Then

$$f^{(m_j-k)}(x) \to g(x),$$

(13) Any other type of convergence will do if it permits term-by-term integration a suitable number of times.

so if we set

$$H_j(x) \equiv \sum_{s=1}^{k} f^{(m_j-s)}(c)\frac{(x-c)^{k-s}}{(k-s)!}, \tag{33}$$

then from (32) follows

$$H_j(x) \to g(x) - G_k(x). \tag{34}$$

Choose k distinct values $x_1, \ldots\ldots, x_k$ on I, all different from c. Substitution into (33) gives a system of k equations in $f^{(m_j-s)}(c)/(k-s)!$, $s = 1, 2, \ldots\ldots, k$, with a non-vanishing Vandermonde determinant. This determinant, moreover, is independent of j, so solving we get

$$f^{(m_j-s)}(c) = \sum_{r=1}^{k} a_{s,r}\, H_j(x_r) \qquad (s = 1, \ldots, k), \tag{35}$$

where the constants $a_{s,r}$ are independent of j. From (34) we see that the right side of (35) has a limit, say a_s, as $j \to \infty$, so

$$\lim_{j\to\infty} f^{(m_j-s)}(c) = a_s \qquad (s = 1, \ldots, k). \tag{36}$$

From (33) and (34) we have

$$\sum_{s=1}^{k} a_s \frac{(x-c)^{k-s}}{(k-s)!} = g(x) - G_k(x). \tag{37}$$

Now $G_k(x)$ is at least of class $C^{(k-1)}$ on I, so

$$a_1 = g^{(k-1)}(x) - \int_c^x g(t)dt;$$

and since $g(x)$ is continuous on I, we may differentiate once more :

$$g^{(k)}(x) = g(x). \tag{38}$$

Since the general solution of the differential equation (38) is an exponential sum of " order " k, it follows that $g(x)$ has the representation (31).

Corollary 1. *If more than one value k satisfies condition* (iii), *let d = g.c.d. of all such k. Then $g(x)$ is of " order " d.*

Proof. The argument of the corollary to Theorem 3 applies here.

Corollary 2. *Under the conditions of Theorem 4, we have uniformly on I,*

$$\lim_{j\to\infty} f^{(m_j-s)}(x) = g^{(k-s)}(x) \qquad (s = 1, 2, \dots, k-1). \quad (39)$$

Proof. Using (36) and (32), we have

$$f^{(m_j-s)}(x) \to \sum_{r=1}^{s} a_r \frac{(x-c)^{s-r}}{(s-r)!} + G_s(x).$$

If we integrate (38) successively from c to x, and use the definition of the functions $G_j(x)$, we get

$$G_s(x) = g^{(k-s)}(x) - \sum_{r=1}^{s} g^{(k-r)}(c) \frac{(x-c)^{s-r}}{(s-r)!};$$

so, uniformly on I,

$$f^{(m_j-s)}(x) \to \sum_{r=1}^{s} \{a_r - g^{(k-r)}(c)\} \frac{(x-c)^{s-r}}{(s-r)!} + g^{(k-s)}(x), (s=1,\dots,k). \quad (40)$$

Now $f^{(m_j-k)}(x) \to g(x)$, since $\{m_j-k\} \subset \{n_j\}$, so taking $s=k$ in (40) we get

$$a_r - g^{(k-r)}(c) = 0, \qquad (r = 1, \dots, k).$$

Thus, (39) holds.

We conclude with a result[14] having some similarity to Theorem 2. However, it restricts $\{x_n\}$ more, and $\{y_n\}$ less, than does Theorem 2.

Theorem 5. *Let $\{a_n\}$ satisfy the conditions* (i) $a_n \neq 0$, $n \geqslant N$; (ii) $\limsup_{n\to\infty} |a_{n+1}/a_n| = \rho < 1$, *and let* $\delta > \frac{\rho}{1-\rho}$ *and* $\phi \geqslant 2$ *be arbitrary but fixed. Then there exists an integer* $V = V[\delta; \phi; \{a_n\}]$ *with the following property: Let,* $\{x_n\}$, $\{y_n\}$ *satisfy the system*

$$\sum_{k=0}^{\infty} a_k x_{n+k} = y_n \qquad (n = 0, 1, \dots), \quad (41)$$

(14) A particular case of this theorem is treated in [4].

where the members of the sequence $\{x_n\}$ *take on only a finite number of distinct values* $s_1, \ldots, s_q$, *with* $q \leqslant \phi$. *Let*

$$\mu = \max |s_i - s_j|, \ \sigma = \min |s_i - s_j|, \qquad (i \neq j).$$

If $\sigma/\mu \geqslant \delta$, *and if more than V members of the sequence* $\{y_n\}$ *are equal, then beginning with some index on, both* $\{x_n\}$ *and* $\{y_n\}$ *are periodic with a period in common.*

The proof bears a resemblance to that of Theorem 2, and moreover follows that given in [4], so it need not be reproduced here.

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ERRATA

(*The Journal of the Indian Mathematical Society, Vol. XV*)

ON UNBOUNDED OPERATORS IN HILBERT SPACE

By M. H. STONE

p. 157, line 25, insert " $\in \mathfrak{M}$ " after " (o, x) "

p. 169, line 23, replace " Cy " by " Cz "

p. 175 line 5, replace " $\mathfrak{D}(A)^*$ " by " $\mathfrak{D}(A^*)$ "

p. 175, line 26, omit " linear " at end of line.

p. 176, line 12, replace " Bx " by " Ax "

p. 178, lines 9, 10, replace " $\mathfrak{N}$ " by " $\mathfrak{M}$ "

p. 186, line 27, replace " $Y_n x$ " by " $Y_n^* x$ "

p. 187, line 4, replace " $S_2 R D_1 x$ " by " $S_2^* R D_1 x$ "
and " any D " by " any D_1 "

p. 187, line 34, insert " x " after " RSD_n "

p. 188, line 23, the last two terms of the equation should read
" $C_y B_w w = B_w C_y w$ "

p. 190, line 23, replace " $\mathfrak{M}$ " by " $\mathfrak{N}$ "

.52

CONNECTION BETWEEN FUNCTIONAL EQUATIONS AND MODULAR RELATIONS, AND FUNCTIONS OF EXPONENTIAL TYPE

By S. BOCHNER

[Received November 6, 1951]

In a recent paper (Some properties of modular relations, *Annals of Math.* **53** (1951), 332-363) we have re-studied the connection between functional equations for zeta functions and modular relations for theta functions from a rather broad approach (broad in a certain direction, that is), and it was pertinent to our approach to introduce a certain class of functions termed *residual.* The description of these functions as fitting into the context was a rather technical one, and an illuminating equivalent description will be given now.

Original Definition. A function $P(x)$ is residual if: (i) it is defined and differentiable in $0 < x < \infty$, and for some $\gamma > 0$ we have $P(x) = O(x^{-\gamma})$ as $x \to 0$ and $P(x) = O(x^{\gamma})$ as $x \to \infty$, so that in the plane of the complex variable $s = \sigma + it$ we may introduce the functions

$$\chi_r(s) = \int_0^1 P(x)\, x^{s-1}\, dx, \tag{1}$$

$$\chi_l(s) = \int_1^\infty P(x)\, x^{s-1}\, dx, \tag{2}$$

the first in a right half-plane, the second in a left half-plane; (ii) these two functions can be continued into one another in a domain D which is the exterior of a bounded closed set S, and (iii) if we denote the joint value of the continued function by $\chi_0(s)$ then we have

$$\lim_{|t| \to \infty} \chi_0(\sigma + it) = 0 \tag{3}$$

uniformly in every finite interval

$$\sigma_1 \leqslant \sigma \leqslant \sigma_2. \tag{4}$$

Now, the role of the residual functions was as follows. We take two Mellin integrals

$$\chi_1(s) = \int_0^\infty x^{s-1}\,\phi(x)\,dx, \tag{5}$$

$$\chi_2(s) = \int_0^\infty x^{s-1}\,\psi(x)\,dx, \tag{6}$$

and we assume that they are absolutely convergent in a right half-plane, that is

$$\int_0^\infty x^{\sigma-1}\,|\phi(x)|\,dx < \infty, \qquad \int_0^\infty x^{\sigma-1}\,|\psi(x)|\,dx < \infty,$$

for $\sigma \geqslant \sigma_0$; with the integrands $\phi(x)$, $\psi(x)$ being differentiable in the open interval $0 < x < \infty$, say.

If the two functions (5), (6) satisfy the functional equation

$$\chi_1(s) = \chi_2(\delta - s) \tag{7}$$

for some

$$\delta > 0$$

in the sense that the two functions

$$\chi_1(s),\ \chi_2(\delta - s) \tag{8}$$

can be continued into one another in a domain D as before, and that for the joint value $\chi(s)$ of the functions (8) we have

$$\lim_{|t|\to\infty} \chi(\sigma + it) = 0$$

uniformly in every interval (4); then the Mellin integrands satisfy the modular relation

$$\phi(x) - x^{-\delta}\,\psi(1/x) = P(x), \tag{9}$$

for some suitable residual function $P(x)$.

Conversely, if two functions $\phi(x)$, $\psi(x)$ satisfy the relation (9) for a suitable residual function $P(x)$, and if the Mellin integrals (5), (6) exist as before, then the integrals satisfy the functional equation (7) as before.

We are now going to give the alternate description.

THEOREM. *A function $P(x)$ in $0 < x < \infty$ is residual if and only if it can be represented as a series*

$$P(x) = \sum_0^\infty \frac{c_n}{n!}(\log 1/x)^n \tag{10}$$

with

$$\gamma_0 = \limsup_{n\to\infty} |c_n|^{1/n} < \infty\,; \tag{11}$$

that is to say, if and only if the function $f(y) = P(e^{-y})$ *is a function of exponential type*

$$f(y) = \sum_0^\infty \frac{c_n y^n}{n!} \tag{12}$$

in the variable y; *and the function* $\chi_0(s)$ *of the definition is the Borel transform*

$$\chi_0(s) = \sum_0^\infty \frac{c_n}{s^{n+1}}\,. \tag{13}$$

Also, if $P(x)$ *appears in a modular relation* (9) *as before, and if we then introduce the function* $\chi(s)$ *as before, then* $\chi_0(s)$ *is the singular part of* $\chi(s)$ *in the sense that* $\chi(s) - \chi_0(s)$ *is an entire function in* s.

Proof. If $P(x)$ is given as a series (10) then property (i) of the definition is a consequence of the exponential character of the function (12). As regards property (ii) we note that for $\sigma > 0$, we have

$$\int_0^1 \left(\log\frac{1}{x}\right)^n x^{s-1}\,dx = \int_0^\infty y^n e^{-sy}\,dy = \frac{1}{s^{n+1}}\,,$$

and for $\sigma > \gamma_0$ this can be summed into

$$\int_0^1 P(x)\,x^{s-1}\,dx = \chi_0(s),$$

where $\chi_0(s)$ is defined by (13). Now, this $\chi_0(s)$ converges for $|s| > \gamma_0$, and it can be easily verified that for $\sigma < -\gamma_0$, we have

$$-\int_1^\infty P(x)\,x^{s-1}\,dx = \chi_0(s),$$

and that (3) holds.

Conversely, let $P^*(x)$ be given residual. The function $\chi_0(s)$ entering its definition is analytic outside a circle and bounded there, and thus has an expansion

$$c_0 + \sum_1^\infty \frac{c_n}{s^{n+1}}$$

with (11) holding. But, it is not hard to see, that the further property (3) implies $c_0 = 0$, and thus $\chi_0(s)$ has the form (13). But now, with these coefficients c_n we can define a function $P(x)$ by the series (10), and the equality

$$\int_0^1 P^*(x)\, x^{s-1}\, dx = \int_0^1 P(x)\, x^{s-1}\, dx \qquad [= \chi_0(s)]$$

will now imply $P(x) = P^*(x)$, by uniqueness of Mellin transforms.

Next, if a simple curve C contains the origin, then

$$\frac{1}{2\pi i}\int_C \frac{x^{-s}}{s^{n+1}}\, ds = \frac{1}{n!}(\log 1/x)^n,$$

and if C contains the circle $|s| = \gamma_0$, then

$$\frac{1}{2\pi i}\int_C \chi_0(s)\, x^{-s}\, ds = P(x), \tag{14}$$

and, furthermore, term-by-term computation of this integral shows that for given $P(x)$, the function $\chi_0(s)$ is uniquely determined by (14) provided it be a Borel transform (13) to start with.

Finally, if $P(x)$ occurs in a modular relation (9) and $\chi(s)$ is then the joint value of the functions (8), then we have (see loc. cit. p. 337)

$$P(x) = \frac{1}{2\pi i}\int_C \chi(s)\, x^{-s}\, dx. \tag{15}$$

However, $\chi(s)$ being analytic outside a circle, it has a Laurent expansion

$$\sum_0^{\infty} \frac{c_n}{s^{n+1}} + \sum_{-\infty}^{-1} \frac{c_n}{s^{n+1}} = \chi_0(s) + \chi(s),$$

where $\chi(s)$ is an entire function. The contribution of the latter to the integral (15) is zero, and by comparing then (15) with (14) we deduce the second half of our theorem as well.

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1952

ON SUMMATION OVER LATTICE POINTS

By T. V. AVADHANI

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1. Introduction. The aim of this paper is to study the Riesz summability of the series

$$\sum_{n=0}^{\infty} n^{\sigma}\, r_k(n, h) \tag{1}$$

and

$$\sum_{n=0}^{\infty} n^{\sigma} J_{\nu}(2\pi\xi\sqrt{n})\, r_k(n, h) \tag{2}$$

for real values of σ, where

$$r_k(n, h) = \sum_{n_1^2+n_2^2+\ldots+n_k^2=n} e^{2\pi i(n_1 h_1+n_2 h_2+\ldots+n_k h_k)}.$$

Bochner and Chandrasekharan [2] studied the series (2) in detail using the theory of spherical summation of multiple Fourier series. Following Landau, Hardy and others we use the Poisson summation formula and its relation to summation over lattice points, and obtain summability indices of the above series for $\sigma > 0$ in case (1), and $[\sigma + \nu/2] \geqslant \nu/2 + 1/4$ in case (2).

It is proved that if $\sigma > 0$ and h is not a lattice point, i.e. $h = (h_1, h_2, \ldots, h_k)$ is not a point all of whose coordinates are integers, $\sum_{n=0}^{\infty} n^{\sigma} r_k(n, h)$ is summable $(R\,;\, n, a)$ for $a > 2\sigma + (k-1)/2$. As regards series (2) it is observed that if $[\sigma + \nu/2] \geqslant \nu/2 + 1/4$, h is not a lattice point and $0 \neq \xi^2 \neq (n_1 + h_1)^2 + \ldots + (n_k + h_k)^2$ for all integral values of $n_1, \ldots, n_k$, $\sum_{n=0}^{\infty} n^{\sigma} J_{\nu}(2\pi\,\xi\sqrt{n})\, r_k\,(n, h)$ is summable $(R\,;\, n,\, a)$ for $a > 2\sigma + k/2 - 1$, while if h is a lattice point it is summable $(R\,;\, n,\, a)$ for $a > 2\sigma + k - 3/2$. Our result

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agrees with that of Bochner and Chandrasekharan when h is a lattice point.

The paper is divided into four sections. In the second section we have collected together some results on Bessel functions and lemmas on Poisson summation formulae which are needed in the later sections. Lemma 2 which is a formula on fractional integrals involving Bessel functions is a generalization of a formula of Dixon and Ferrar [3]. Lemmas 5 and 6 deal with fractional integrals involving a product of two Bessel functions and are believed to be new. In the third section we study the summability of the series (1) and (2). Lemma 3 is made use of for the summability of the series (1) and Lemmas 5 and 6 for the summability of the series (2). We have also added theorems on absolute summability of the two series. The convexity theorem on the abscissa of summability of Dirichlet series is made use of in the case of series (2), while no use is made of it in the case of series (1). In the concluding section we study the summability of the series (1) and (2) for negative values of σ using a Tauberian theorem of Ananda Rau.

2. Preliminary Lemmas.

LEMMAS ON BESSEL FUNCTIONS. The following properties of Bessel functions [4] are needed for our discussion. As usual a Bessel function of order μ is denoted by $J_\mu(x)$.

$$J_\mu(x) = \sum_{p=0}^{\infty} \frac{(-1)^p \, (\frac{1}{2}x)^{2p+\mu}}{\Gamma(p+1)\,\Gamma(p+\mu+1)}, \quad \mu > -1, \tag{2.1}$$

$$J_{\frac{1}{2}}(x) = (2/\pi x)^{1/2} \sin x \text{ and } J_{-\frac{1}{2}}(x) = (2/\pi x)^{1/2} \cos x. \tag{2.2}$$

For a fixed $\mu > -1$,

$$x^{1/2} J_\mu(x) \sim \cos x \sum_{n=0}^{m-1} A_n x^{-n} + \sin x \sum_{n=0}^{m-1} B_n x^{-n} + O(x^{-m}), \tag{2.3}$$

where A_n and B_n are suitable constants. If $\mu < x/2$

$$J_\mu(x) = O(x^{-1/2}), \qquad x \to \infty. \tag{2.4}$$

For fixed values of a, b, ν, δ and $\nu > -1$, $\delta > -1$, and $a \neq b$

$$t J_\nu(at)\, J_\delta(bt) \sim \psi(t) + O(t^{-m}), \tag{2.5}$$

for large values of t (say $t > N$), where

$$\psi(t) = \cos(a+b)t \sum_{n=0}^{m-1} A_n t^{-n} + \cos(a-b)t \sum_{n=0}^{m-1} B_n t^{-n} + \\ + \sin(a+b)t \sum_{n=0}^{m-1} C_n t^{-n} + \sin(a-b)t \sum_{n=0}^{m-1} D_n t^{-n},$$

where A_n, B_n, C_n and D_n are suitable constants. For all x, and $\mu \geqslant 0$

$$|J_\mu(x)| \leqslant 1. \tag{2.6}$$

For α and $\beta > -1$,

$$\int_0^R \left(1 - \frac{t^2}{R^2}\right)^\alpha t^{\beta+1} J_\beta(2\pi a t)\, dt = \tfrac{1}{2} \frac{\Gamma(\alpha+1)}{(\pi a)^{\alpha+1}} R^{\alpha+\beta+1} \times \\ \times J_{\alpha+\beta+1}(2\pi a R). \tag{2.7}$$

If $\rho_h^2 = h_1^2 + h_2^2 + \ldots + h_k^2 \neq 0$,

$$\int_P e^{2\pi i(u_1 h_1 + \ldots + u_k h_k)}\, du_1 \ldots du_k = \frac{2\pi}{\rho_h^{k/2-1}} \int_0^R t^{k/2} J_{k/2-1}(2\pi \rho_h t)\, dt, \tag{2.8}$$

where P is the k-dimensional sphere of radius R, centre being the origin. If $h_1, \ldots, h_k$ are all simultaneously zero, i.e. if $\rho_h^2 = 0$,

$$\int_P e^{2\pi i(u_1 h_1 + \ldots + u_k h_k)}\, du_1 \ldots du_k = \frac{2(\pi)^{k/2}}{\Gamma(k/2)} \int_0^R t^{k-1}\, dt, \tag{2.9}$$

where P has the same meaning as in (2.8).

Lemma 1. *For large positive real values of* x

$${}_1F_2\left(\begin{matrix}\alpha \\ \beta, \gamma\end{matrix}\, x e^{i\pi}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\alpha)}{\Gamma(n+\beta)\Gamma(n+\gamma)\Gamma(n+1)} x^n \\ = \frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)\,\Gamma(\gamma-\alpha)} x^{-\alpha} + O\left(x^{(\alpha-\beta-\gamma)/2+1/4}\right) + O(x^{-\alpha-1}). \tag{2.10}$$

See [3, 177 (2.12)].

LEMMA 2. *For fixed values of* α, β, δ, *such that* $\alpha > -1$, $\delta > -1$, $\beta + \delta + 1 > 0$, *and* $R \geqslant 0$, $a > 0$,

$$\int_0^R \left(1 - \frac{t^2}{R^2}\right)^\alpha t^\beta J_\delta(2\pi a t)\,dt = \frac{\Gamma(\alpha+1)}{2} R^{\beta+\delta+1} (\pi a)^\delta \times$$

$$\times\, {}_1F_2\left(\frac{\beta+\delta+1}{2}, \delta+1, \alpha + \frac{\beta+\delta+3}{2}, -\pi^2 a^2 R^2\right) \quad (2.11)$$

$$= \frac{\Gamma\{(\beta+\delta+1)/2\}}{2\Gamma\{(-\beta+\delta+1)/2\}} \frac{1}{(\pi a)^{\beta+1}} + O\left(\frac{1}{a^{\alpha+3/2}} \frac{1}{R^{\alpha-\beta+\frac{1}{2}}}\right) +$$

$$+ O\left(\frac{1}{a^{\beta+3}} \frac{1}{R^2}\right), \quad (2.12)$$

as $aR \to \infty$.

PROOF. Since $\delta > -1$, using (2.1) the integral on the left side of (2.11) turns out to be

$$\int_0^R \left(1 - \frac{t^2}{R^2}\right)^\alpha t^\beta \sum_{p=0}^{\infty} \frac{(-1)^p (\pi a t)^{2p+\delta}}{\Gamma(p+1)\,\Gamma(p+\delta+1)}\,dt$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p (\pi a)^{2p+\delta}}{\Gamma(p+1)\,\Gamma(p+\delta+1)} \int_0^R \left(1 - \frac{t^2}{R^2}\right)^\alpha t^{\beta+2p+\delta}\,dt. \quad (2.13)$$

Putting $t^2 = R^2 u$ in (2.13) and integrating we obtain (2.11). Using Lemma 1 in (2.11) we easily obtain (2.12).

LEMMA 3. *For* $\alpha \geqslant 0$ *and fixed* $a > 0$

$$\lim_{R\to\infty} \int_a^R \left(1 - \frac{t^2}{R^2}\right)^\alpha f(t)\,dt = \int_a^\infty f(t)\,dt$$

whenever the latter exists.

LEMMA 4. *For* $c \neq 0$ *and* $\lambda > 0$,

$$\lim_{R\to\infty} \int_\lambda^R \left(1 - \frac{t^2}{R^2}\right)^\alpha t^\beta \,{\cos \atop \sin}\,(ct)\,dt,$$

exists (a) *for every* $\alpha \geqslant 0$ *if* $\beta < 0$, *and* (b) *for* $\alpha > \beta$ *if* $\beta \geqslant 0$.

PROOF. (a). By Lemma 3

$$\lim_{R\to\infty} \int_\lambda^R \left(1 - \frac{t^2}{R^2}\right)^\alpha t^\beta \,{\cos \atop \sin}\,(ct)\,dt = \int_\lambda^\infty t^\beta \,{\cos \atop \sin}\,(ct)\,dt,$$

whenever the right side exists. But we know that the above integral on the right converges for all values of $\beta < 0$.

(b). $\lim_{R\to\infty}\int_{\lambda}^{R}\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta}\frac{\cos}{\sin}(ct)\,dt$

$$=\lim_{R\to\infty}\left\{\int_0^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta}\frac{\cos}{\sin}(ct)\,dt-\int_0^{\lambda}\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta}\frac{\cos}{\sin}(ct)\,dt\right\}.$$

As $c \neq 0$ by (2.2) we have

$$=\lim_{R\to\infty}\left(\frac{\pi c}{2}\right)^{1/2}\int_0^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta+\frac{1}{2}} J_{\pm\frac{1}{2}}(ct)\,dt\,-$$

$$-\lim_{R\to\infty}\int_0^{\lambda}\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta}\frac{\cos}{\sin}(ct)\,dt. \quad (2.14).$$

Since $\beta \geqslant 0$, using Lemma 2 we have

$$\int_0^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta+\frac{1}{2}} J_{\pm\frac{1}{2}}(ct)\,dt=\text{a constant}+O(1/R^{\alpha-\beta})+O(1/R^2). \quad (2.15).$$

But

$$\lim_{R\to\infty}\int_0^{\lambda}\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta}\frac{\cos}{\sin}(ct)\,dt=\int_0^{\lambda} t^{\beta}\frac{\cos}{\sin}(ct)\,dt. \quad (2.16).$$

The Lemma follows from (2.14) in view of (2.15) and (2.16).

LEMMA 5. *If* $\alpha \geqslant 0, \nu > -1, \delta > -1, \beta+\nu+\delta > -1,$ *and* $a \neq b$,

$$\int_0^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta} J_{\nu}(at) J_{\delta}(bt)\,dt=\text{a constant}+O(1/R^{\alpha-\beta+1})+O(1/R^2), \quad (2.17)$$

for large values of R.

PROOF. Let m be the least positive integer greater than β.

$$\lim_{R\to\infty}\int_0^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta} J_{\nu}(at) J_{\delta}(bt)dt=\lim_{R\to\infty}\int_0^{\lambda}\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta} J_{\nu}(at) J_{\delta}(bt)dt$$

$$+\lim_{R\to\infty}\int_{\lambda}^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{\beta} J_{\nu}(at) J_{\delta}(bt)\,dt, \quad (2.18)$$

where λ is a fixed positive number greater than N of (2.5). The first integral on the right side of (2.18) is

$$\int_0^{\lambda} t^{\beta} J_{\nu}(at) J_{\delta}(bt)\,dt. \quad (2.19)$$

And (2.19) is convergent as $\beta + \nu + \delta > -1$. Using (2.5) for $J_\nu(at)J_\delta(bt)$ in the integrand of the second integral on the right side of (2.18) we obtain that it is

$$\lim_{R\to\infty}\int_\lambda^R \left(1-\frac{t^2}{R^2}\right)^a t^{\beta-1}\,\psi(t)\,dt + \lim_{R\to\infty}\int_\lambda^R \left(1-\frac{t^2}{R^2}\right)^a t^{\beta-1}\,O(t^{-m})\,dt, \tag{2.20}$$

where $\psi(t)$ has the same meaning as in (2.5). But the second integral on the right side of (2.20) tends to a limit as $R\to\infty$, since it is

$$O\left[\lim_{R\to\infty}\int_\lambda^R \left(1-\frac{t^2}{R^2}\right)^a t^{\beta-m-1}\,dt\right] = O\left(\int_\lambda^\infty t^{\beta-m-1}\,dt\right) \text{ by Lemma 3,} \tag{2.21}$$

$$= O(1), \text{ since } \beta < m.$$

The first integral on the right side of (2.20) contains only a finite number of terms, a typical term of which is, but for a suitable multiplying constant,

$$\int_\lambda^R \left(1-\frac{t^2}{R^2}\right)^a t^{\beta-1-r}\,{\cos \atop \sin}(ct)\,dt, \tag{2.22}$$

where $c \neq 0$ and r may take any one of the values $0, 1, \ldots, m-1$. By Lemma 4 (a), (2.22) converges when $r = m-1$, whereas from (2.15) and (2.16) we observe that for $r = 0, 1, \ldots, m-2$,

$$(2.22) \text{ is a constant} + O(1/R^{a-\beta+1+r}) + O(1/R^2).$$

So the first integral on the right side of (2.20) is

$$\text{a constant} + O(1/R^{a-\beta+1}) + O(1/R^2). \tag{2.23}$$

We observe that (2.17) follows from (2.19), (2.21) and (2.23).

LEMMA 6. *If* $a > -1$, $\nu > -1$, $\delta > -1$, $\beta + \nu - \delta - 1$ *is an even integer* $2l \geqslant 0$, $R \geqslant 0$, $k > 1$, $a > 0$, $b > ka$, *then*

$$\int_0^R \left(1-\frac{t^2}{R^2}\right)^a t^\beta J_\nu(2\pi at)\,J_\delta(2\pi bt)\,dt$$

$$= \frac{1}{2}\frac{R^{\beta-a}}{(\pi b)^{a+1}}\sum_{\mu=0}^{l}(-1)^\mu\binom{l}{\mu}\frac{1}{(\pi bR)^\mu}\sum_{\lambda=0}^{\infty}\frac{\Gamma(a+\mu+\lambda+1)}{\Gamma(\lambda+1)}\left(\frac{a}{b}\right)^\lambda \times$$

$$\times J_{a+\mu+\lambda+\delta+1}(2\pi bR)\,J_{\lambda+\nu}(2\pi aR), \tag{2.24}$$

$$= O\left(\frac{1}{b^{a+3/2}}\cdot\frac{1}{R^{a-\beta+1}}\right), \tag{2.25}$$

uniformly for $b > ka$, $R > a^{-1}$, where the constant in O depends on α, ν, δ, β, a and k.

PROOF. Writing $t^{2l} = \{R^2 - (R^2 - t^2)\}^l$ and expanding as a binomial we have

$$t^{2l} = R^{2l} \sum_{\mu=0}^{l} (-1)^{\mu} \binom{l}{\mu} \left(1 - \frac{t^2}{R^2}\right)^{\mu}. \qquad (2.26)$$

Since l is a non-negative integer we have only a finite number of terms in (2.26). Using (2.26) we observe that

$$\int_0^R \left(1 - \frac{t^2}{R^2}\right)^{\alpha} t^{\beta} J_{\nu}(2\pi at)\, J_{\delta}(2\pi bt)\, dt$$

$$= \int_0^R \left(1 - \frac{t^2}{R^2}\right)^{\alpha} t^{2l-\nu+\delta+1} J_{\nu}(2\pi at)\, J_{\delta}(2\pi bt)\, dt$$

$$= R^{-2\alpha} \sum_{\mu=0}^{l} (-1)^{\mu} \binom{l}{\mu} R^{2l-2\mu} \int_0^R (R^2 - t^2)^{\alpha+\mu} t^{-\nu+\delta+1} \times$$

$$\times J_{\nu}(2\pi at)\, J_{\delta}(2\pi bt)\, dt. \qquad (2.27)$$

As $\nu > -1$, writing the infinite series for $J_{\nu}(2\pi at)$ in the integrand on the right side of (2.27) we have

$$\sum_{p=0}^{\infty} \frac{(-1)^p (\pi a)^{2p+\nu}}{\Gamma(p+1)\,\Gamma(p+\nu+1)} \int_0^R (R^2 - t^2)^{\alpha+\mu} t^{2p+\delta+1} J_{\delta}(2\pi bt)\, dt, \qquad (2.28)$$

where p is a non-negative integer. So writing an expression similar to (2.26) for t^{2p} also, (2.28) turns out to be

$$\sum_{p=0}^{\infty} \frac{(-1)^p (\pi a)^{2p+\nu}}{\Gamma(p+1)\,\Gamma(p+\nu+1)} \sum_{\lambda=0}^{p} (-1)^{\lambda} \binom{p}{\lambda} R^{2p-2\lambda} \times$$

$$\times \int_0^R (R^2 - t^2)^{\alpha+\mu+\lambda} t^{\delta+1} J_{\delta}(2\pi bt)\, dt. \qquad (2.29)$$

As $\delta > -1$, using (2.7) the integral in (2.29) will be

$$\frac{1}{2} \frac{\Gamma(\alpha+\mu+\lambda+1)}{(\pi b)^{\alpha+\mu+\lambda+1}} R^{\alpha+\mu+\lambda+\delta+1} J_{\alpha+\mu+\lambda+\delta+1}(2\pi b R). \qquad (2.30)$$

Substituting (2.30) for the integral in (2.29) and interchanging the order of summation for p and λ in (2.29) we obtain

$$\tfrac{1}{2} R^{-2\alpha} \sum_{\mu=0}^{l} (-1)^{\mu} \binom{l}{\mu} \frac{R^{\alpha+\delta+2l-\nu+1}}{(\pi b)^{\alpha+1} (\pi b R)^{\mu}} \sum_{\lambda=0}^{\infty} \frac{\Gamma(\alpha+\mu+\lambda+1)}{\Gamma(\lambda+1)} \left(\frac{a}{b}\right)^{\lambda} \times$$

$$\times J_{\alpha+\mu+\lambda+\delta+1}(2\pi bR)\, J_{\lambda+\nu}(2\pi aR). \qquad (2.31)$$

Using (2.6) we note that the infinite series in (2.31) converges absolutely. Therefore (2.24) follows from (2.31). Write the convergent infinite series in (2.31) as $\Sigma_1 + \Sigma_2$, say, where $\Sigma_1 = \sum_{\lambda=0}^{[\frac{1}{2}aR-1]}$ and $\Sigma_2 = \sum_{\lambda=[\frac{1}{2}aR]}^{\infty}$. Using (2.4),

$$\Sigma_1 = O\left(\frac{1}{b^{\frac{1}{2}}R}\right)\left[\frac{\Gamma\{\alpha+\mu+1\}}{1} + \frac{\Gamma\{\alpha+\mu+2\}}{\Gamma(2)}\left(\frac{a}{b}\right) + \dots + \right.$$

$$\left. + \frac{\Gamma\{\alpha+\mu+[\frac{1}{2}aR]\}}{\Gamma\{[\frac{1}{2}aR]\}}\left(\frac{a}{b}\right)^{[\frac{1}{2}aR-1]}\right],$$

$$= O\left(\frac{1}{b^{\frac{1}{2}}R}\right)\left[1 + \binom{\alpha+\mu+1}{1}\left(\frac{a}{b}\right) + \dots + \infty\right],$$

$$= O(1/b^{\frac{1}{2}}R), \text{ since } a/b < k < 1.$$

Using (2.6) in Σ_2 we observe

$$|\Sigma_2| = \left(\frac{a}{b}\right)^{[\frac{1}{2}aR]}\frac{\Gamma\{\alpha+\mu+[\frac{1}{2}aR]+1\}}{\Gamma\{[\frac{1}{2}aR]+1\}} \times$$

$$\times\left\{1 + \frac{\alpha+\mu+[\frac{1}{2}aR]+1}{[\frac{1}{2}aR]+1}\left(\frac{a}{b}\right) + \dots + \infty\right\}$$

$$< \left(\frac{a}{b}\right)^{[\frac{1}{2}aR]}\frac{\Gamma\{\alpha+\mu+[\frac{1}{2}aR]+1\}}{\Gamma\{[\frac{1}{2}aR]+1\}} \times$$

$$\times\left\{1 + \binom{\alpha+\mu+1}{1}\left(\frac{a}{b}\right) + \dots + \infty\right\}$$

$$< \left(\frac{a}{b}\right)^{[\frac{1}{2}aR]}\frac{\Gamma\{\alpha+\mu+[\frac{1}{2}aR]+1\}}{\Gamma\{[\frac{1}{2}aR]+1\}}.\frac{1}{(1-a/b)^{\alpha+\mu+1}}. \tag{2.32}$$

Using Stirling's formula we know that for a fixed α,

$$\frac{\Gamma\{\alpha+x\}}{\Gamma\{x\}} = O(x^{\alpha})$$

as $x \to \infty$. Using the above relation in (2.32) we have that (2.32)

$$= \left(\frac{a}{b}\right)^{[\frac{1}{2}aR]}\left([\tfrac{1}{2}aR]+1\right)^{\alpha+\mu} = O\left(\frac{1}{b^{\frac{1}{2}}R}\right).$$

For large values of b and R, the convergent infinite series in (2.31) is

$$O(1/b^{\frac{1}{2}}R). \tag{2.33}$$

From (2.24) and (2.33) we observe that (2.25) follows easily.

LEMMAS ON LATTICE POINTS. In what follows we use the following notation:

$n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k$, take only integral values from $-\infty$ to $+\infty$ and $h_1, h_2, \ldots, h_k$ and $u_1, u_2, \ldots, u_k$ take only real values. We say the point $h = (h_1, h_2, \ldots, h_k)$ is a lattice point if $h_i = n_i$.

$n = n_1^2 + n_2^2 + \ldots + n_k^2$; $\rho^2_{(m,h)} = (m_1 + h_1)^2 + \ldots + (m_k + h_k)^2$ and ξ is a real number not equal to any $\rho_{(m,h)}$.

$$r_k(n,h) = \sum_{n_1^2 + \ldots + n_k^2 = n} e^{2\pi i(n_1 h_1 + \ldots + n_k h_k)}.$$

LEMMA 7.

$$\sum_{n_1, \ldots, n_k, n \neq 0} \frac{1}{(n_1^2 + \ldots + n_k^2)^\sigma} \text{ is convergent if } \sigma > k/2.$$

This is a well-known result.

LEMMA 8. *If P is the k-dimensional sphere of radius R and*

$$g(u) = g(u, \delta) = \sum_{-\infty}^{+\infty} e^{-(m-u)^2/\delta^2}$$

and $\psi(u_1, \ldots, u_k)$ is any continuous function, then

$$\lim_{\delta \to 0} \frac{1}{(\sqrt{\pi}\,\delta)^k} \int_P g(u_1) \ldots g(u_k)\, \psi(u_1, \ldots, u_k)\, du_1 \ldots du_k = \Sigma' \psi(n_1, \ldots, n_k),$$

where the dash implies that the lattice points on the boundary of the sphere are counted with a weight 1/2 and the summation extends over the lattice points of P.

However it is to be noted that if R^2 is chosen to be non-integral there will be no lattice points on the boundary of the sphere.

LEMMA 9.

$$\sum_{\substack{-\infty \\ m_1, \ldots, m_k}}^{+\infty} \exp\{-[(m_1 - u_1)^2 + \ldots + (m_k - u_k)^2]/\delta^2\} = (\sqrt{\pi}\,\delta)^k \times$$

$$\times \sum_{\substack{-\infty \\ m_1, \ldots, m_k}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2 + \ldots + m_k^2) + 2\pi i(m_1 u_1 + \ldots + m_k u_k)\}$$

We have only to take the product of $g(u_1),\ldots, g(u_k)$ and employ the well-known formula

$$\sum_{-\infty}^{+\infty} \exp\{-(m-u)^2/\delta^2\} = (\sqrt{\pi}\,\delta) \sum_{-\infty}^{+\infty} \exp\{-\pi^2\delta^2 m^2 + 2\pi i m u\}.$$

Taking Lemmas 8 and 9 together we have

LEMMA 10.

$$\Sigma' \psi(n_1, \ldots, n_k) = \lim_{\delta\to 0} \sum_{\substack{-\infty \\ m_1,\ldots,m_k}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\} \times$$

$$\times \int_P \exp\{2\pi i(m_1u_1+\ldots+m_ku_k)\}\, \psi(u_1,\ldots,u_k)\, du_1\ldots du_k,$$

where $\psi(u_1\ldots, u_k)$ is any continuous function and the dash has the same meaning as in Lemma 8.

LEMMA 11.

(i) *If $\alpha \geqslant 0$ and $\beta \geqslant 0$*

$$\sum_{0<n\leqslant R^2} \left(1-\frac{n}{R^2}\right)^{\alpha} n^{\beta} r_k(n,h) = \lim_{\delta\to 0} \sum_{\substack{-\infty \\ m_1,\ldots,m_k}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\} \times$$

$$\times\, \Gamma(\alpha+1)(\pi)^{k/2} R^{2\beta+k}\, {}_1F_2(\beta+k/2, k/2, \alpha+\beta+k/2+1, -\pi^2\rho_{(m,h)}^2R^2) \tag{2.34}$$

if h is not a lattice point, whereas

$$\sum_{0<n\leqslant R^2} \left(1-\frac{n}{R^2}\right)^{\alpha} n^{\beta} r_k(n) = \frac{(\pi)^{k/2}\,\Gamma(\alpha+1)\Gamma(\beta+k/2)}{\Gamma(k/2)\Gamma(\alpha+\beta+k/2+1)} R^{2\beta+k} +$$

$$+ \lim_{\delta\to 0} \sum_{\substack{-\infty \\ m_1,\ldots,m_k,\, m\neq 0}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\, \Gamma(\alpha+1)(\pi)^{k/2} R^{2\beta+k} \times$$

$$\times\, {}_1F_2(\beta+k/2, k/2, \alpha+\beta+k/2+1, -\pi^2\rho_{(m,h)}^2 R^2). \tag{2.35}$$

(ii) *If $\alpha \geqslant 0$, $\beta \geqslant -\nu/2$ and $\nu > -1$, for $\xi \neq 0$,*

$$\sum_{0<n\leqslant R^2} \left(1-\frac{n}{R^2}\right)^{\alpha} n^{\beta} J_\nu(2\pi\xi\sqrt{n})\, r_k(n,h) =$$

$$\lim_{\delta\to 0} \sum_{\substack{-\infty \\ m_1,\ldots,m_k}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\} \times$$

$$\times \int_0^R \frac{2\pi}{\rho_{(m,h)}^{k/2-1}} \left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{2\beta+k/2} J_\nu(2\pi\xi t)\, J_{k/2-1}(2\pi\rho_{(m,h)}t)\, dt \tag{2.36}$$

if h is not a lattice point, whereas

$$\sum_{0<n\leqslant R^2}\left(1-\frac{n}{R^2}\right)^a n^\beta J_\nu(2\pi\,\xi\sqrt{n})\,r_k(n)$$
$$=\frac{2(\pi)^{k/2}}{\Gamma(k/2)}\int_0^R\left(1-\frac{t^2}{R^2}\right)^a t^{2\beta+k-1}J_\nu(2\pi\,\xi\,t)\,dt\;+$$
$$+\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1\ldots m_k,m\neq 0}}^{+\infty}\exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\times$$
$$\times\int_0^R\frac{2\pi}{\rho_m^{k/2-1}}\left(1-\frac{t^2}{R^2}\right)^a t^{2\beta+k/2}J_\nu(2\pi\,\xi\,t)\,J_{k/2-1}(2\pi\,\rho_m t)\,dt. \quad (2.37)$$

PROOF. (i) Choose

$$\psi(u_1,\ldots,u_k)=\left[1-\frac{\Sigma u_l^2}{R^2}\right]^a(\Sigma u_l^2)^\beta\exp\{2\pi i\,(\Sigma u_l h_l)\}$$

in Lemma 10. We observe that

$$\sum_{0<n\leqslant R^2}\left(1-\frac{n}{R^2}\right)^a n^\beta r_k(n,h)=\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1,\ldots,m_k}}^{+\infty}\exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\times$$
$$\times\int_P\left(1-\frac{\Sigma u_l^2}{R^2}\right)^a(\Sigma u_l^2)^\beta\exp\{2\pi i\,(\Sigma u_l\,\overline{m_l+h_l})\}\,du_1\ldots du_k. \quad (2.38)$$

If h is not a lattice point, introducing polar coordinates and using (2.8), we observe that the right side of (2.38) turns out to be

$$\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1,\ldots,m_k}}^{+\infty}\exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\times$$
$$\times\int_0^R\frac{2\pi}{\rho_{(m,h)}^{k/2-1}}\left(1-\frac{t^2}{R^2}\right)^a t^{2\beta+k/2}J_{k/2-1}(2\pi\,\rho_{(m,h)}t)\,dt. \quad (2.39)$$

Using (2.11) for the integral in (2.39) we easily obtain (2.34) from (2.39). If h is a lattice point (2.35) can be written as

$$\sum_{0<n\leqslant R^2}\left(1-\frac{n}{R^2}\right)^a n^\beta r_k(n)=\int_P\left(1-\frac{\Sigma u_l^2}{R^2}\right)^a(\Sigma u_l^2)^\beta du_1\ldots du^k+$$
$$+\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1,\,.,m_k,\,m\neq 0}}^{+\infty}\exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\times$$
$$\times\int_P\left(1-\frac{\Sigma u_l^2}{R^2}\right)^a(\Sigma u_l^2)^\beta\exp\{2\pi i\,(\Sigma u_l m_l)\}\,du_1\ldots du_k. \quad (2.40)$$

Using (2.9) and introducing polar coordinates the first term of (2.40) turns out to be

$$\frac{2(\pi)^{k/2}}{\Gamma(k/2)}\int_0^R t^{2\beta+k-1}\left(1-\frac{t^2}{R^2}\right)^{\alpha} dt. \tag{2.41}$$

Putting $t^2 = R^2 u$ in (2.41) and integrating we have

$$\frac{(\pi)^{k/2}}{\Gamma(k/2)}\frac{\Gamma(\alpha+1)\ \Gamma(\beta+k/2)}{\Gamma(\alpha+\beta+k/2+1)} R^{2\beta+k}. \tag{2.42}$$

Introducing polar coordinates and using (2.8), the second part of (2.40) turns out to be

$$\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1,\ldots,m_k,\ m\neq 0}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\times$$
$$\times\frac{2\pi}{\rho_m^{k/2-1}}\int_0^R\left(1-\frac{t^2}{R^2}\right)^{\alpha} t^{2\beta+k/2} J_{k/2-1}(2\pi\rho_m t)\,dt. \tag{2.43}$$

Using (2.11) for the integral in (2.43) we observe that (2.35) follows easily from (2.42) and (2.43).

(ii) Choosing

$$\psi(u_1,\ldots,u_k)=\left(1-\frac{\Sigma u_l^2}{R^2}\right)^{\alpha}(\Sigma u_l^2)^{\beta} J_\nu(2\pi\xi\sqrt{\Sigma u_l^2})\exp\{2\pi i\,(\Sigma u_l h_l)\}$$

in Lemma 10 and proceeding exactly as in (i) we obtain (2.36) and (2.37).

Lemma 12. *If $\sigma_{(k)}$ denotes the abscissa of summability $(R\,;n,k)$ of the Dirichlet series $\Sigma a_n/n^{\sigma}$ then $\sigma_{(k)}$ is a convex function of k.*

This is a well-known theorem.

3. Summability theorems for positive values of σ.

The Series $\sum_{n=0}^{\infty} n^{\sigma} r_k(n,h)$. (3.1)

When h is not a lattice point, if $\sigma>0$ and $\alpha\geqslant 0$, from (2.34) we have

$$\sum_{0<n\leqslant R^2}{}' \left(1-\frac{n}{R^2}\right)^{\alpha} n^{\sigma} r_k(n,h)$$
$$=\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1,\ldots,m_k}}^{+\infty}\exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\,\Gamma(\alpha+1)\,(\pi)^{k/2} R^{2\sigma+k}\times$$
$$\times\ {}_1F_2(\sigma+k/2,k/2,\alpha+\sigma+k/2+1,-\pi^2\rho^2_{(m,h)}R^2)$$

$$= \sum_{\substack{-\infty \\ m_1, \dots, m_k}}^{+\infty} \lim_{\delta \to 0} \exp\{-\pi^2 \delta^2 (m_1^2 + \dots + m_k^2)\} \Gamma(a+1) (\pi)^{k/2} R^{2\sigma+k} \times$$

$$\times {}_1F_2 (\sigma + k/2, k/2, a + \sigma + k/2 + 1, -\pi^2 \rho^2_{(m,h)} R^2)$$

$$= \sum_{\substack{-\infty \\ m_1, \dots, m_k}}^{+\infty} \Gamma(a+1) (\pi)^{k/2} R^{2\sigma+k} \times$$

$$\times {}_1F_2 (\sigma + k/2, k/2, a + \sigma + k/2 + 1, -\pi^2 \rho^2_{(m,h)} R^2), \quad (3.2)$$

provided this converges. For large values of $\rho_{(m,h)} R$, using Lemma 1, (3.2) is

$$\frac{\Gamma(\sigma + k/2)}{\Gamma(-\sigma)} \sum \frac{1}{\{\pi \rho_{(m,h)}\}^{2\sigma+k}} + O\Big(\sum \frac{1}{\rho_{(m,h)}^{a+(k+1)/2}} \frac{1}{R^{a-2\sigma-k/2+1/2}}\Big) +$$

$$+ O\Big(\sum \frac{1}{\rho_{(m,h)}^{2\sigma+k+2}} \frac{1}{R^2}\Big). \quad (3.3)$$

Using Lemma 7 for $\sigma > 0$ we observe that if $a > (k-1)/2$

$$\left.\begin{array}{l} \sum \frac{1}{\rho_{(m,h)}^{2\sigma+k}} \\ \sum \frac{1}{\rho_{(m,h)}^{a+(k+1)/2}} \\ \sum \frac{1}{\rho_{(m,h)}^{2\sigma+k+2}} \end{array}\right\} < \infty \quad (3.4)$$

It follows from (3.3) and (3.4) that the expression on the right side of (3.2) is, for $\sigma > 0$,

$$\frac{\Gamma(\sigma + k/2)}{\Gamma(-\sigma)} \sum \frac{1}{\{\pi \rho_{(m,h)}\}^{2\sigma+k}} + O\left(\frac{1}{R^{a-2\sigma-k/2+1/2}}\right) + O\Big(\frac{1}{R^2}\Big). \quad (3.5)$$

But we know that for positive integral values of σ, $1/\Gamma(-\sigma) = 0$. Hence we note that (3.5) is

$$\left.\begin{array}{ll} \frac{\Gamma(\sigma + k/2)}{\Gamma(-\sigma)} \sum \frac{1}{\{\pi \rho_{(m,h)}\}^{2\sigma+k}} + o(1) & \text{for positive non-integral } \sigma \text{ and } a > 2\sigma + (k-1)/2, \\ o(1), & \text{for positive integral } \sigma, \text{ and } a > 2\sigma + (k-1)/2, \\ O(1), & \text{for positive } \sigma \text{ and } a = 2\sigma + (k-1)/2 \end{array}\right\}. \quad (3.6)$$

We therefore have

THEOREM 1. *If h is not a lattice point, for all positive values of σ, the series $\sum_{n=0}^{\infty} n^{\sigma} r_k(n, h)$ is summable $(R; n, a)$ to*

$$\frac{\Gamma(\sigma + k/2)}{\Gamma(-\sigma)} \frac{1}{\pi^{2\sigma+k}} \sum \left\{\frac{1}{(n_1 + h_1)^2 + \dots + (n_k + h_k)^2}\right\}^{2\sigma+k},$$

and in particular to zero if σ is a positive integer, for $a > 2\sigma + (k-1)/2$ and is bounded $(R; n, a)$ for $a = 2\sigma + (k-1)/2$.

When h is a lattice point, if σ is positive and $a \geqslant 0$ we have from (2.35)

$$\sum_{0<n\leqslant R^2} \left(1 - \frac{n}{R^2}\right)^a n^{\sigma} r_k(n) = \frac{(\pi)^{k/2}}{\Gamma(k/2)} \frac{\Gamma(a+1)\,\Gamma(\sigma+k/2)}{\Gamma(a+\sigma+k/2+1)} R^{2\sigma+k}$$

$$+ \lim_{\delta\to 0} \sum_{\substack{-\infty \\ m_1,\dots,m_k,\, m\neq 0}}^{+\infty} \exp\{-\pi^2 \delta^2 (m_1^2 + \dots + m_k^2)\}\, \Gamma(a+1)\, (\pi)^{k/2} R^{2\sigma+k} \times$$

$$\times\, {}_1F_2 (\sigma + k/2, k/2, a + \sigma + k/2 + 1, -\pi^2 \rho_m^2 R^2) \quad (3.7)$$

Proceeding similarly as for (3.2) we observe that for large values of $\rho_m R$ the second term on the right side of (3.7) will be

$$\frac{\Gamma(\sigma + k/2)}{\Gamma(-\sigma)} \sum \frac{1}{\{\pi \rho_m\}^{2\sigma+k}} + O\left(\frac{1}{R^{a-2\sigma-k/2+1/2}}\right) + O\left(\frac{1}{R^2}\right), \quad (3.8)$$

for positive values of σ, if $a > (k-1)/2$. Using (3.8) we observe that (3.7) reduces to

$$\frac{(\pi)^{k/2}}{\Gamma(k/2)} \cdot \frac{\Gamma(a+1)\,\Gamma(\sigma+k/2)}{\Gamma(a+\sigma+k/2+1)} R^{2\sigma+k} + \frac{\Gamma(\sigma+k/2)}{\Gamma(-\sigma)} \sum_{\substack{-\infty \\ m_1,\dots,m_k,\, m\neq 0}}^{+\infty} \frac{1}{\{\pi \rho_m\}^{2\sigma+k}} +$$

$$+ O\left(\frac{1}{R^{a-2\sigma-k/2+1/2}}\right) + O\left(\frac{1}{R^2}\right). \quad (3.9)$$

THEOREM 2. *For $\sigma > 0$ and $a > (k-1)/2$,*

$$\sum_{0<n\leqslant R^2} \left(1 - \frac{n}{R^2}\right)^a n^{\sigma} r_k(n) = \frac{(\pi)^{k/2}}{\Gamma(k/2)} \frac{\Gamma(a+1)\,\Gamma(\sigma+k/2)}{\Gamma(a+\sigma+k/2+1)} R^{2\sigma+k} +$$

$$+ \frac{\Gamma(\sigma+k/2)}{\Gamma(-\sigma)} \sum_{\substack{-\infty \\ m_1,\dots,m_k,\, m\neq 0}}^{+\infty} \frac{1}{\{\pi \rho_m\}^{2\sigma+k}} + O\left(\frac{1}{R^{a-2\sigma-k/2+1/2}}\right) + O\left(\frac{1}{R^2}\right).$$

Let us now consider the absolute summability of the series (3.1). From (3.5) we observe by putting $R^2 = x$ that for $a > (k+1)/2$ and $\sigma > 0$

$$\sum_{0<n\leqslant x} \left(1-\frac{n}{x}\right)^{a-1} n^{\sigma+1} r_k(n,h) = C + O\left(\frac{1}{x^{a/2-\sigma-k/4-5/4}}\right) + O\left(\frac{1}{x}\right), \tag{3.10}$$

where C is a constant and h is not a lattice point. Using (3.10) we have

$$a\int^{+\infty} \frac{1}{x^2}\left|\sum\left(1-\frac{n}{x}\right)^{a-1} n^{\sigma+1} r_k(n,h)\right| dx$$

$$\leqslant \int^{+\infty} \frac{|c|}{x^2} dx + A\int^{+\infty} \frac{dx}{x^{a/2-\sigma-k/4+3/4}} + B\int^{+\infty} \frac{dx}{x^3}, \tag{3.11}$$

where A and B are constants. The first and the third integrals on the right side of (3.11) converge and the second integral also converges for positive values of σ, if $a > 2\sigma + (k+1)/2$. We have the following theorem.

THEOREM 3. *For all positive values of σ, if h is not a lattice point the series $\sum_{n=0}^{\infty} n^\sigma r_k(n,h)$ is summable $|n, a|$ for $a > 2\sigma + (k+1)/2$.*

THE SERIES $\sum_{n=0}^{\infty} n^\sigma J_\nu(2\pi\xi\sqrt{n})\, r_k(n,h)$. (3.12)

Let us first consider the summability of the series (3.12) for non-negative integral values of l, where $l = (\sigma + \nu/2)$. We note that $l \geqslant 0$ if $\sigma \geqslant -\nu/2$.

When h is not a lattice point and when $a \geqslant 0$, $\xi \neq 0$ and $\sigma \geqslant -\nu/2$, $\nu > -1$, we have from (2.36)

$$\sum_{0<n\leqslant R^2} \left(1-\frac{n}{R^2}\right)^a n^\sigma J_\nu(2\pi\xi\sqrt{n})\, r_k(n,h)$$

$$= \sum_{0<n\leqslant R} \left(1-\frac{n}{R^2}\right)^a n^{l-\nu/2} J_\nu(2\pi\xi\sqrt{n})\, r_k(n,h)$$

$$= \lim_{\delta\to 0} \sum_{\substack{-\infty \\ m_1,\ldots m_k}}^{+\infty} \exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\} \times$$

$$\times \int_0^R \frac{2\pi}{\rho_{(m,h)}^{k/2-1}} \left(1 - \frac{t^2}{R^2}\right)^a t^{2l-\nu+k/2} J_\nu(2\pi \xi t) J_{k/2-1}(2\pi \rho_{(m,h)} t)\, dt$$

$$= \sum_{m_1 \ldots m_k}^{+\infty}{}_{-\infty} \lim_{\delta \to 0} \exp\{-\pi^2 \delta^2(m_1^2 + \ldots + m_k^2)\} \times$$

$$\times \int_0^R \frac{2\pi}{\rho_{(m,h)}^{k/2-1}} \left(1 - \frac{t^2}{R^2}\right)^a t^{2l-\nu+k/2} J_\nu(2\pi \xi t) J_{k/2-1}(2\pi \rho_{(m,h)} t)\, dt$$

$$= \sum_{m_1 \ldots m_k}^{+\infty}{}_{-\infty} \int_0^R \frac{2\pi}{\rho_{(m,h)}^{k/2-1}} \left(1 - \frac{t^2}{R^2}\right)^a t^{2l-\nu+k/2} \times J_\nu(2\pi \xi t) J_{k/2-1}(2\pi \rho_{(m,h)} t)\, dt$$

$$= \Sigma_1 + \Sigma_2, \text{ say,} \tag{3.13}$$

where Σ_1 stands for the summation over such values of $m_1, m_2, \ldots m_k$ for which $\xi > \rho_{(m,h)}$ and Σ_2 stands for the summation over the rest of the values of $m_1, m_2, \ldots m_k$. If $\xi \neq \rho_{(m,h)}$ we observe that Σ_2 stands for the summation over the values of $m_1, m_2, \ldots m_k$ for which $\xi < \rho_{(m,h)}$. Σ_1 has only a finite number of terms and by (2.17) each term is

$$\text{a constant} + O\left(\frac{1}{R^{a-2l+\nu-k/2+1}}\right) + O(1/R^2).$$

We therefore obtain that

$$\Sigma_1 = \text{a constant} + O\left(\frac{1}{R^{a-2l+\nu-k/2+1}}\right) + O(1/R^2). \tag{3.14}$$

Using (2.25) we observe that each term in Σ_2 is

$$O\left(\frac{1}{\rho_{(m,h)}^{a+(k+1)/2}} \cdot \frac{1}{R^{a-2l+\nu-k/2+1}}\right)$$

and therefore

$$\Sigma_2 = O\left(\frac{1}{R^{a-2l+\nu-k/2+1}} \Sigma_2 \frac{1}{\rho_{(m,h)}^{a+(k+1)/2}}\right)$$

$$= O\left(\frac{1}{R^{a-2l+\nu-k/2+1}}\right) \quad \text{if } a > (k-1)/2, \tag{3.15}$$

since by Lemma 7, $\sum_{-\infty}^{+\infty} 1/\rho_{(m,h)}^{a+k+1/2}$ converges if $a > (k-1)/2$. We obtain from (3.14) and (3.15) that if $\xi \neq \rho_{(m,h)}$ and if $a > (k-1)/2$, (3.13) is

$$\text{a constant} + O\left(\frac{1}{R^{a-2l+\nu-k/2+1}}\right) + O(1/R^2). \tag{3.16}$$

Hence we observe that (3.13) is

a constant + o(1) for $(l-\nu/2)\geqslant\frac{1}{4}$ and $a>2l-\nu+k/2-1$;
O(1) for $(l-\nu/2)>\frac{1}{4}$ and $a=2l-\nu+k/2-1$.

The series (3.12) is therefore summable $(R;\ n,\ a)$ for $a>2l-\nu+k/2-1$ whenever $\sigma=(l-\nu/2)\geqslant\frac{1}{4}$ and is bounded $(R;\ n,\ a)$ for $a=2l-\nu+k/2-1$ whenever $\sigma=(l-\nu/2)>\frac{1}{4}$ and l takes only non-negative integral values. By Lemma 12, since the abscissa of summability is a convex function of l, the above result holds good for every $l\geqslant 0$ integral or otherwise.

We see that if further $2l-2\nu+k-2$ is an even integer $\geqslant 0$, the series (3.12) is summable to zero. For Σ_1 contains only a finite number of terms and by Lemma 6 each term in Σ_1 is $O(1/R^{a-2l+\nu-k/2+1})$ if $2l-2\nu+k-2$ is an even integer $\geqslant 0$. So,

$$\Sigma_1=O\left(\frac{1}{R^{a-2l+\nu-k/2+1}}\right) \tag{3.17}$$

if $(\nu-k/2)$ is an integer $\leqslant(l-1)$. From (3.15) and (3.17) we observe that (3.13) is $O\left(\frac{1}{R^{a-2l+\nu-k/2+1}}\right)$ or (3.13) is

$$\left.\begin{array}{ll} o(1) & \text{if } (l-\nu/2)\geqslant\frac{1}{4} \text{ and } (\nu-k/2) \text{ is an} \\ & \quad\text{integer} \leqslant(l-1) \text{ for } a>2l-\nu+k/2-1 \\ O(1) & \text{if } (l-\nu/2)>\frac{1}{4} \text{ for } a=2l-\nu+k/2-1 \end{array}\right\}. \tag{3.18}$$

We therefore have the following theorems.

THEOREM 4. *If* $0\neq\xi\neq\rho_{(m,h)}$ *and* $\nu>-1$ *and if* h *is not a lattice point, the series* $\sum_{n=0}^{\infty} n^{\sigma} J_{\nu}(2\pi\xi\sqrt{n})\, r_k(n,h)$ *is summable* $(R;\ n,\ a)$ *for* $a>2\sigma+k/2-1$ *if* $[\sigma+\nu/2]\geqslant\nu/2+\frac{1}{4}$ *and is bounded* $(R;\ n,\ a)$ *for* $a=2\sigma+k/2-1$ *if* $[\sigma+\nu/2]>\nu/2+\frac{1}{4}$.

THEOREM 5. *For* $0\neq\xi\neq\rho_{(m,h)}$, *if* h *is not a lattice point, the series* $\sum_{n=0}^{\infty} n^{\sigma}\dfrac{J_{\beta+k/2}(2\pi\xi\sqrt{n})}{n^{\beta/2+k/4}}\, r_k(n,\ h)$ *is summable to zero* $(R;\ n,\ a)$ *for* $a>2\sigma-\beta-1$ *and* $(k-1)/2$ *for every pair of positive integral values of the type* $\beta=n$ *and* $\sigma=n+1, n+2,\ldots$, *where* $n=0, 1, \ldots$.

If $(\sigma+\nu/2)\geqslant 0, a\geqslant 0, \xi\neq 0, \nu>-1$ we have from (2.37)

$$\sum_{0<n\leqslant R^2}\left(1-\frac{n}{R^2}\right)^a n^\sigma J_\nu(2\pi\xi\sqrt{n})\, r_k(n)$$

$$=\frac{2(\pi)^{k/2}}{\Gamma(k/2)}\int_0^R\left(1-\frac{t^2}{R^2}\right)^a t^{2\sigma+k-1} J_\nu(2\pi\xi t)\,dt+$$

$$+\lim_{\delta\to 0}\sum_{\substack{-\infty\\ m_1,\ldots,m_k,\ m\neq 0}}^{+\infty}\exp\{-\pi^2\delta^2(m_1^2+\ldots+m_k^2)\}\times$$

$$\times\int_0^R\frac{2\pi}{\rho_m^{k/2-1}}\left(1-\frac{t^2}{R^2}\right)^a t^{2\sigma+k/2} J_\nu(2\pi\xi t)\, J_{k/2-1}(2\pi\rho_m t)\,dt. \qquad (3.19)$$

Using (2.12) we observe that for large values of R, the first term in the right side of (3.19) is

$$\frac{(\pi)^{k/2}\,\Gamma\{\sigma+(k+\nu)/2\}}{\Gamma(k/2)\,\Gamma\{-\sigma+1+(\nu-k)/2\}}\,\frac{1}{(\pi\,\xi)^{2\sigma+k}}+O\left(\frac{1}{R^{a-2\sigma-k+3/2}}\right)+$$

$$+O(1/R^2). \quad (3.20)$$

We observe that if $a>(k-1)/2$ and if $\xi\neq\rho_m$ using (3.16) the second term on the right side of (3.19) is

$$\text{a constant}+O\left(\frac{1}{R^{a-2\sigma-k/2+1}}\right)+O(1/R^2). \qquad (3.21)$$

It follows from (3.20) and (3.21) that for $a>(k-1)/2$, (3.19) is

$$\text{a constant}+O\left(\frac{1}{R^{a-2\sigma-k+3/2}}\right)+O(1/R^2). \qquad (3.22)$$

From (3.22) it follows that (3.19) is
a constant $+\ o(1)$ for $a>2\sigma+k-3/2$ if $[\sigma+\nu/2]\geqslant\nu/2+\frac{1}{4}$,
$O(1)$ for $a=2\sigma+k-3/2$ if $[\sigma+\nu/2]>\nu/2+\frac{1}{4}$.
We have the following theorem.

THEOREM 6. *If* $0\neq\xi\neq\rho_m$ *and* $\nu>-1$, *the series* $\sum_{n=0}^{\infty} n^\sigma J_\nu(2\pi\xi\sqrt{n})\, r_k(n)$ *is summable* $(R; n, a)$ *for* $a>2\sigma+k-3/2$ *whenever* $[\sigma+\nu/2]\geqslant\nu/2+\frac{1}{4}$ *and is bounded* $(R;\ n,\ a)$ *for* $a=2\sigma+k-3/2$ *if* $[\sigma+\nu/2]>\nu/2+\frac{1}{4}$.

Let us now consider the absolute summability of the series (3.12). For $a > (k+1)/2$ by putting $R^2 = x$ we obtain from (3.16) that

$$\sum_{0<n\leqslant x} \left(1-\frac{n}{x}\right)^{a-1} n^{\sigma+1} J_\nu(2\pi\,\xi\,\sqrt{n})\, r_k(n,h)$$
$$= C + O\left(\frac{1}{x^{a/2-\sigma-k/4-1}}\right) + O(1/x), \tag{3.23}$$

where C is a constant, and from (3.22) it follows that if $a > (k+1)/2$

$$\sum_{0<n\leqslant x} \left(1-\frac{n}{x}\right)^{a-1} n^{\sigma+1} J_\nu(2\pi\,\xi\,\sqrt{n})\, r_k(n)$$
$$= C' + O\left(\frac{1}{x^{a/2-\sigma-k/2-3/4}}\right) + O(1/x), \tag{3.24}$$

where C' is a constant. Using (3.23) we have

$$a\int^\infty \frac{1}{x^2}\left|\sum\left(1-\frac{n}{x}\right)^{a-1} n^{\sigma+1} J_\nu(2\pi\,\xi\,\sqrt{n})\, r_k(n,h)\right| dx$$
$$\leqslant \int^\infty \frac{|c|}{x^2}\,dx + A\int^\infty \frac{dx}{x^{a/2-\sigma-k/4+1}} + B\int^\infty \frac{dx}{x^3}, \tag{3.25}$$

where A and B are constants. The first and the third integrals on the right side of (3.25) converge, and for $[\sigma+\nu/2] \geqslant \nu/2+\frac{1}{4}$, the second integral converges if $a > 2\,\sigma + k/2$. Using (3.24) we obtain that

$$a\int^\infty \frac{1}{x^2}\left|\sum\left(1-\frac{n}{x}\right)^{a-1} n^{\sigma+1} J_\nu(2\pi\,\xi\,\sqrt{n})\, r_k(n)\right| dx$$
$$\leqslant \int^\infty \frac{|c'|}{x^2}\,dx + D\int^\infty \frac{dx}{x^{a/2-\sigma-k/2+5/4}} + E\int^\infty \frac{dx}{x^3}, \tag{3.26}$$

where D and E are constants. The first and the third integrals on the right side of (3.26) converge whereas for $[\sigma+\nu/2] \geqslant \nu/2+\frac{1}{4}$, the second integral converges if $a > 2\,\sigma + k - \frac{1}{2}$. We have therefore the following theorem.

Theorem 7. *For all values of* $[\sigma+\nu/2] \geqslant \nu/2+\frac{1}{4}$ *if* $0 \neq \xi \neq \rho_{(m,h)}$ *and* $\nu > -1$, *the series* $\sum\limits_{n=0}^{\infty} n^\sigma J_\nu(2\pi\,\xi\,\sqrt{n})\, r_k(n,h)$

is summable $|n, a|$ *for* $a > 2\sigma + k/2$ *whereas the series* $\sum_{n=0}^{\infty} n^{\sigma} J_{\nu}(2\pi\,\xi\,\sqrt{n})\, r_k(n)$ *is summable* $|n, a|$ *for* $a > 2\sigma + k - \frac{1}{2}$ *provided* $0 \neq \xi \neq \rho_m$.

4. Summability theorems for negative values of σ.

LEMMA 13. *If* Σa_n *is summable* $(R\,;\, n, \lambda)$, $\lambda \geqslant 0$ *and if* $a_n = O(n^p)$ *then* $\Sigma a_n/n^{\sigma}$ *is summable* $(R; n, a)$ *for* $a > \lambda - \sigma\, \dfrac{\lambda+1}{p+1}$ *for* $0 < \sigma \leqslant \lambda \dfrac{p+1}{\lambda+1}$.

This is a special case of a theorem of Ananda Rau [1].

Applying Lemma 13 to Theorem 1 with $a_n = n^{\epsilon/2}\, r_k(n, h)$ and noting that $r_k(n, h) = O(n^{k/2-1+\epsilon})$ we obtain the following theorem.

THEOREM 8. *If* h *is not a lattice point the series* $\sum_{n=1}^{\infty} \dfrac{r_k(n, h)}{n^{\sigma}}$ *is summable* $(R; n, a)$ *for* $a > \dfrac{k-1}{2} - \sigma\left(1 + \dfrac{1}{k}\right)$ *if* $0 < \sigma \leqslant \dfrac{k(k-1)}{2(k+1)}$.

From Theorem 8 it follows that $\sum_{n=1}^{\infty} \dfrac{r_k(n, h)}{n^{\sigma}}$ converges if $\sigma > k(k-1)/2(k+1)$. Let λ be the greatest integer less than $\nu/2 + \frac{1}{4}$. With $\sigma = \lambda - \nu/2 + 1$ it follows from Theorem 4 that $\Sigma n^{\lambda-\nu/2+1}\, r_k(n, h)\, J_{\nu}(2\,\pi\,\xi\,\sqrt{n})$ is summable $(R;\, n,\, a)$ for $a > 2\lambda - \nu + k/2 + 1$. Applying Lemma 13 to this result with

$$a_n = n^{\lambda-\nu/2+1} J_{\nu}(2\,\pi\,\xi\,\sqrt{n})\; r_k(n, h)$$

which is $O(n^{\lambda-\nu/2+k/2-\frac{1}{4}+\epsilon})$ we obtain the following theorem.

THEOREM 9. *If* h *is not a lattice point and* $0 \neq \xi \neq \rho_{(m, h)}$, $\nu > -1$, *the series* $\sum_{n=1}^{\infty} \dfrac{J_{\nu}(2\pi\,\xi\,\sqrt{n})\, r_k(n, h)}{n^{\sigma}}$ *is summable* $(R;\, n,\, a)$ *for* $a > -2\sigma + \dfrac{k}{2} - 1 + \dfrac{k-1}{2} \cdot \dfrac{\sigma + \lambda - \nu/2 + 1}{\lambda - \nu/2 + k/2 + \frac{3}{4}}$ *if*

$$-\lambda + \nu/2 - \tfrac{3}{4} < \sigma + \tfrac{1}{4} \leqslant \frac{k}{2} - \frac{\lambda - \nu/2 + k/2 + \frac{3}{4}}{2\lambda - \nu + k/2 + 2}.$$

With $\sigma = \lambda - \nu/2 + 1$ it follows from Theorem 6 that

$$\sum_{n=0}^{\infty} n^{\lambda-\nu/2+1} J_\nu(2\pi \xi \sqrt{n})\, r_k(n)$$

is summable $(R; n, a)$ for $a > 2\lambda - \nu + k + \frac{1}{2}$. Applying Lemma 13 to this result with $a_n = n^{\lambda-\nu/2+1} J_\nu(2\pi \xi \sqrt{n})\, r_k(n)$ we obtain the following theorem.

THEOREM 10. *For* $0 \neq \xi \neq \rho_m$ *and* $\nu > -1$ *the series* $\sum_{n=1}^{\infty} \frac{J_\nu(2\pi \xi \sqrt{n})\, r_k(n)}{n^\sigma}$ *is summable* $(R; n, a)$ *for* $a > -2\sigma + k - 3/2$ *if* $-\lambda + \nu/2 - \frac{3}{4} < \sigma + \frac{1}{4} \leqslant \frac{k-1}{2}$.

Regarding the absolute summability of the series (3.1) and (3.12) for negative values of σ we easily obtain the following theorem from Theorems 8, 9 and 10 just in the same manner as Theorems 3 and 7 from Theorems 1, 4 and 6.

THEOREM 11. (i) *If h is not a lattice point the series* $\sum_{n=1}^{\infty} \frac{r_k(n,h)}{n^\sigma}$ *is summable* $|n, a|$ *for* $a > \frac{k+1}{2} - \sigma\left(1 + \frac{1}{k}\right)$ *if* $0 < \sigma \leqslant k/2$.

(ii) *For all values of* $-\lambda + \nu/2 - \frac{3}{4} < \sigma + \frac{1}{4} \leqslant k/2$ *the series* $\sum_{n=1}^{\infty} \frac{J_\nu(2\pi \xi \sqrt{n})\, r_k(n,h)}{n^\sigma}$ *is summable* $|n, a|$ *for*

$$a > k/2 - 2\sigma + \frac{k-1}{2}\, \frac{\sigma + \lambda - \nu/2 + 1}{\lambda - \nu/2 + k/2 + 3/4} \quad \text{if } 0 \neq \xi \neq \rho_{(m,\, h)},$$

whereas the series $\sum_{n=1}^{\infty} \frac{J_\nu(2\pi \xi \sqrt{n})\, r_k(n)}{n^\sigma}$ *is summable* $|n, a|$ *for* $a > k - \frac{1}{2} - 2\sigma$ *if* $0 \neq \xi \neq \rho_m$.

It follows from Theorem 11 that the series $\sum_{n=1}^{\infty} \frac{r_k(n,h)}{n^\sigma}$ converges absolutely if $\sigma > k/2$ whereas the series $\sum_{n=1}^{\infty} J_\nu(2\pi \xi \sqrt{n}) \times$

$\times\, r_k(n, h)/n^\sigma$ and $\sum_{n=1}^{\infty} J_\nu(2\pi\,\xi\,\sqrt{n})\, r_k(n)/n^\sigma$ converge absolutely if $(\sigma + \frac{1}{4}) > k/2$.

Employing (2.2) it follows from Theorem 9 that the series $\sum_{n=1}^{\infty} \frac{\cos(\sqrt{n}\,t)\, r_k(n,h)}{n^\sigma}$ is summable $(R; n, a)$ for $a > \frac{k-1}{2} - \sigma\left(1 + \frac{1}{k}\right)$ if $0 < \sigma \leqslant \frac{k(k-1)}{2(k+1)}$ whereas the series $\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}\,t)\, r_k(n, h)}{n^\sigma}$ is summable $(R; n, a)$ for $a > \frac{(k-1)(k+2)}{2(k+1)} - \sigma\left(1 + \frac{2}{k+1}\right)$ if $-\frac{1}{2} < \sigma \leqslant \frac{(k-1)(k+2)}{2(k+3)}$ provided $0 \neq t^2 \neq (n_1 + h_1)^2 + \dots + (n_k + h_k)^2$. We therefore deduce that provided $0 \neq t^2 \neq (n_1 + h_1)^2 + \dots + (n_k + h_k)^2$ the series $\sum_{n=1}^{\infty} \frac{\cos(\sqrt{n}\,t)\, r_k(n,h)}{n^\sigma}$ converges if $\sigma > \frac{k(k-1)}{2(k+1)}$ whereas the series $\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}\,t)\, r_k(n,h)}{n^\sigma}$ converges if $\sigma > \frac{(k-1)(k+2)}{2(k+3)}$.

It follows from Theorem 10 that the series $\sum_{n=1}^{\infty} \frac{\cos(\sqrt{n}\,t)\, r_k(n)}{n^\sigma}$ is summable $(R; n, a)$ for $a > -2\sigma + k - 1$ if $0 < \sigma \leqslant (k-1)/2$ whereas the series $\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}\,t)\, r_k(n)}{n^\sigma}$ is summable $(R; n, a)$ for $a > -2\sigma + k - 1$ if $-\frac{1}{2} < \sigma \leqslant (k-1)/2$ provided $0 \neq t^2 \neq n_1^2 + \dots + n_k^2$. We therefore deduce that the series $\sum_{n=1}^{\infty} \frac{\cos(\sqrt{n}\,t)\, r_k(n)}{n^\sigma}$ as well as the series $\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}\,t)\, r_k(n)}{n^\sigma}$ converge if $\sigma > (k-1)/2$ provided $0 \neq t^2 \neq n_1^2 + \dots + n_k^2$.

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INEQUALITIES BETWEEN SERIES AND INTEGRALS INVOLVING ENTIRE FUNCTIONS

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1. Let $f(z)$ be an entire function of positive exponential type c; Plancherel and Pólya [7] showed that if $f(x)$ belongs to L^p on the real axis (where $p > 0$) then the series

$$\sum_{-\infty}^{\infty} |f(\lambda_n)|^p \tag{1.1}$$

converges provided that $\{\lambda_n\}$ is an increasing real sequence such that

$$\lambda_{n+1} - \lambda_n \geqslant 2\delta > 0. \tag{1.2}$$

(Added in proof: See also Nikolski [6 a].)

I shall extend this result by showing that

$$\int_{-\infty}^{\infty} \phi\{\,|f(x)|\,\}\, dx < \infty\ , \tag{1.3}$$

implies

$$\sum_{-\infty}^{\infty} \phi\{e^{-c\delta}|f(\lambda_n)|\} < \infty\ , \tag{1.4}$$

when $\{\lambda_n\}$ satisfies (1.2), if $\phi(t)$ is a non-negative non-decreasing convex function of $\log t$. The factor $e^{-c\delta}$ cannot in general be omitted: consider, for example, $\phi(t) = \log^+ t$ and $f(z) = (1 + z^{-1}) \sin \pi z/2$. If $f(z)$ is merely of exponential type in a right half-plane, it is easy to make the inference from (1.3) to (1.4) (integral and sum now extending over $(0, \infty)$) if $\phi(t) = t^p$, $p > 1$, but somewhat harder when $\phi(t) = t^p$, $0 < p \leqslant 1$; and I have not been able to do it for as general a $\phi(t)$ as those which can be used for entire functions.

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The converse inference, that $\Sigma\phi\{|f(\lambda_n)|\} < \infty$ implies $\int_{-\infty}^{\infty} \phi\{H|f(x)|\}\,dx < \infty$ for a certain positive H, is more difficult. Plancherel and Pólya [7] established it for $\lambda_n = n$, $c \leqslant \pi$ if $p > 1$, $c < \pi$ if $0 < p \leqslant 1$; Duffin and Schaeffer [3] established it for $\phi(t) = t^2$, and I [1] for a class of $\phi(t)$ including all $t^p, p > 0$, under the hypotheses $|\lambda_n - n| \leqslant L$, $c < \pi$, and (1.2).

As Plancherel and Pólya observed, the convergence of (1.4) is easily obtained if one can show that (1.3) implies a bound for

$$\int_{-\infty}^{\infty} \phi\{e^{-c|y|}|f(x+iy)|\}\,dx, \qquad |y| > 0, \qquad (1.5)$$

and our main problem is to obtain such a bound. This problem can be attacked in various ways. Our assumptions on $\phi(t)$ imply that the integrand of (1.5) is subharmonic in the upper (or lower) half-plane. Now there are theorems which provide an estimate for the integral of a subharmonic function along an interior curve when its integral along the boundary of a region is known; the use of these theorems has, however, two disadvantages. In the first place it does not apply to as general a function $\phi(t)$ as I wish to consider (although it does apply to $\phi(t) = t^p$); in the second place, since the region is unbounded it requires some additional information about the growth of $f(z)$, such as would be furnished by the boundedness of $f(x)$ (rather less would do). Now in fact the convergence of $\int_{-\infty}^{\infty} \phi\{|f(x)|\}\,dx$ implies that $f(\mu_n)$ is bounded for some sequence $\{\mu_n\}$ with (say) $|n - \mu_n| < \frac{1}{2}$, and this, by a theorem of Duffin and Schaeffer [2] implies the boundedness[2] of $f(x)$; but to argue in this way involves an appeal to a deeper theorem than the one we are trying to prove, and it seems desirable to avoid this. Much the same objections apply to attacking the problem by representing the function by means of the

2. Note, however, for a general ϕ, that $f(x) \to 0$; consider, for example, $\phi(t) = \log^+ t$ and $|f(x)| < 1$.

Poisson integral for a half-plane. I shall use the method of Plancherel and Pólya, based on the Phragmén-Lindelöf principle; this yields a stronger result, that (1.3) implies a bound for (1.5) when $f(z)$ is merely assumed regular and of exponential type in the upper half-plane. (In this case $f(x)$ does not have to be bounded. The Poisson integral method yields a bound for (1.5) with a larger number than c in the exponent). For functions which are of exponential type in a right half-plane the method of Plancherel and Pólya does not give enough information and it is necessary to use one of the other methods, with its attendant disadvantages.

2. The following notations will be used throughout. R is a region $x \geqslant A$, $y \geqslant 0$, to be interpreted as the upper half-plane when $A = -\infty$; $f(z)$ is regular in R; $\phi(t)$ is non-negative, non-decreasing, and a convex (non-concave) function of $\log t$, with $\phi(0+) = 0$ but $\phi(t)$ not identically zero, so that $\phi\{|f(t)|\}$ is subharmonic in R [8, 15]. (We could replace $|f(z)|$ throughout by a non-negative function of class PL, i.e. one whose logarithm is subharmonic.) For given finite a and b, $\psi(z)$ is defined by

$$\psi(z) = \int_a^b \phi\{|f(z+s)|\}\, ds, \tag{2.1}$$

with integration along a line-segment parallel to the real axis. We denote by S a region in which $\psi(z)$ is defined. If the circle $|z - z_0| < \delta$ is in S, we have

$$\psi(z_0) = \int_a^b \phi\{|f(z_0+s)|\}\, ds \leqslant \int_a^b \left\{\frac{1}{2\pi}\int_0^{2\pi} \phi\{|f(z_0 + \delta e^{i\theta} + s)|\} d\theta\right\} ds$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \psi(z_0 + \delta e^{i\theta})\, d\theta,$$

since $\phi\{|f(z+s)|\}$ is subharmonic, so that $\psi(z)$ is subharmonic in S.

We say that a function $G(z)$ defined in R is of (exponential) type c in R if $G(z) = O(e^{d|z|})$, uniformly as $z \to \infty$ in R, for each $d > c$.

3. Lemma 1. *If R is $(x \geqslant a, y \geqslant 0)$, S is $(x \geqslant 0, y \geqslant 0)$; if $f(z)$ is of finite type c in R; and if $\psi(x) \leqslant M$ and $\psi(iy) \leqslant N$ $(0 \leqslant x < \infty, 0 \leqslant y < \infty)$, then*

$$\psi(z) \leqslant \max(M, N), \qquad z \in S. \tag{3.1}$$

Take $\epsilon > 0$, $a' < a$, and consider the function

$$G(z) = |f(z)| \exp\{-\epsilon |z - a'|^{\sigma} \cos \sigma(\theta + \gamma)\},$$

where $1 < \sigma < 2$, $-\pi < 2\sigma\gamma < \pi(1 - \sigma)$; $\exp\{\ldots\}$ is the absolute value of a branch of $\exp\{-\epsilon[(z - a')e^{i\gamma}]^{\sigma}\}$ which is regular in R.

In R we have, uniformly for $z \to \infty$,

$$G(z) = O\{\exp[c'|z| - \epsilon|z - a'|^{\sigma} \cos \sigma(\tfrac{1}{2}\pi + \gamma)]\}, \quad c' > c, \tag{3.2}$$

and $G(z) \leqslant |f(z)|$. If $\psi_{\epsilon}(z)$ is the integral in (2.1) with $G(z)$ instead of $|f(z)|$, we have for $z \in S$ (because ϕ is non-decreasing) $\psi_{\epsilon}(z) \leqslant \psi(z)$. Therefore $\psi_{\epsilon}(x) \leqslant M$, $\psi_{\epsilon}(iy) \leqslant N$, for $x \geqslant 0, y \geqslant 0$.

Let z_0 be any interior point of S. Let S_B be the intersection of S and the circle $|z| \leqslant B$, $B > |z_0|$, and let C be the intersection of S and the circumference $|z| = B$. By (3.2), $G(z + s) \to 0$ in R, uniformly for $a \leqslant s \leqslant b$; hence (since $\phi(0+) = 0$),

$$\sup_{z \in C} \psi_{\epsilon}(z) \to 0, \qquad z \to \infty.$$

Since $\psi_{\epsilon}(z)$ is subharmonic in S, its maximum in S_B occurs on the boundary, and (for large B) not on C (unless $M = N = 0$); hence $\psi_{\epsilon}(z_0) \leqslant \max(M, N)$. Letting $\epsilon \to 0$ we obtain (3.1).

Lemma 2. *Under the hypotheses of Lemma 1, there is a K independent of $f(z)$ such that*

$$\phi\{e^{-cy}|f(x + iy + s)|\}ds \leqslant K, \qquad z \in S. \tag{3.3}$$

Take $\epsilon > 0$ and apply Lemma 1 to

$$H(z) = |e^{i(c+\epsilon)z} f(z)| = e^{-(c+\epsilon)y}|f(x + iy)|. \tag{3.4}$$

We have $H(x) = |f(x)|$, $x \geqslant a$, and $H|(x + iy)| \leqslant e^{-(c+\epsilon)y + c|x+iy|}$, $z \in R$. Hence if

$$\psi_{\epsilon}(z) = \int_a^b \phi\{H(z + s)\}\, ds, \tag{3.5}$$

we have $\psi_\varepsilon(x) \leqslant M, x \geqslant 0$, and

$$N_\varepsilon = \sup_{y \geqslant 0} \psi_\varepsilon(iy) \leqslant \int_a^b \phi(e^{c|s|})\, ds = N', \tag{3.6}$$

say. The conclusion of Lemma 1 is then

$$\psi_\varepsilon(z) \leqslant \max(M, N_\varepsilon) \leqslant \max(M, N') = K, \tag{3.7}$$

and since K is independent of ϵ we obtain (3.3) by letting $\epsilon \to 0$.

4. By enlarging R to the half-plane $y \geqslant 0$, we can improve Lemma 2.

LEMMA 3. *If $f(z)$ is of type c in $y \geqslant 0$ and $\psi(x) \leqslant M$, $-\infty < x < \infty$, then*

$$\int_a^b \phi\{e^{-cy}|f(x+iy+s)|\}\, ds \leqslant M, \qquad y \geqslant 0, \tag{4.1}$$

and here a or b or both may be infinite.

Define $H(z)$, $\psi_\varepsilon(z)$, N_ε by (3.4), (3.5), (3.6), respectively. For $x \geqslant 0$, $y \geqslant 0$, we have by (3.7)

$$\psi_\varepsilon(z) \leqslant \max(M, N_\varepsilon). \tag{4.2}$$

In the same way (or by making the change of variable $x' = -x$) we see that (4.2) also holds for $x \leqslant 0$, $y \geqslant 0$.

Now $H(iy+s) \to 0$ as $y \to \infty$, uniformly for $a \leqslant s \leqslant b$; and so $\psi_\varepsilon(iy) \to 0$ as $y \to \infty$. Consequently $N_\varepsilon = \psi_\varepsilon(iy_0)$ for some finite y_0. This implies that $N_\varepsilon \leqslant M$. For, if $N_\varepsilon > M$, we should have $y_0 > 0$ and the subharmonic function $\psi_\varepsilon(z)$ would assume its maximum at an interior point of the upper half plane, and so would be the constant N_ε, which is excluded since $\psi_\varepsilon(x) \leqslant M$. Thus $\psi_\varepsilon(z) \leqslant M$ for $y \geqslant 0$ and by letting $\epsilon \to 0$ we have (4.1).

The last clause of Lemma 3 follows since if, for example, $b = \infty$, we can apply what has already been proved with any finite b and then let $b \to \infty$ in (4.1). We cannot extend Lemma 2 in the same way because K in (3.3) depends on a and b.

5. We can now prove our main theorem.

THEOREM 1. *If $f(z)$ is an entire function of exponential type c and*

$$\int_{-\infty}^{\infty} \phi(|f(s)|)\, ds \leqslant M, \tag{5.1}$$

then

$$\sum_{-\infty}^{\infty} \phi(e^{-c\delta}|f(\lambda_n)|) \leqslant 2\, M/(\pi\, \delta), \tag{5.2}$$

if $\{\lambda_n\}$ is a real increasing sequence such that

$$\lambda_{n+1} - \lambda_n \geqslant 2\,\delta > 0.$$

Since $\phi(e^{-c\delta}|f(z)|)$ is subharmonic, we have

$$\begin{aligned}
\pi\,\delta^2\,\phi(e^{-c\delta}|f(\lambda_n)|) &\leqslant \iint_{s^2+y^2\leqslant\delta^2} \phi\{e^{-c\delta}|f(\lambda_n+s+iy)|\}\, ds\, dy \\
&\leqslant \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \phi\{e^{-c\delta}|f(\lambda_n+s+iy)|\}\, ds\, dy \\
&\leqslant \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \phi\{e^{-c|y|}|f(\lambda_n+s+iy)|\}\, ds\, dy \\
&\leqslant \int_{-\delta}^{\delta} dy \int_{\lambda_n-\delta}^{\lambda_n+\delta} \phi\{e^{-c|y|}|f(s+iy)|\}\, ds.
\end{aligned}$$

Hence for any N, since no two intervals $(\lambda_n - \delta, \lambda_n + \delta)$ overlap,

$$\begin{aligned}
\pi\,\delta^2 \sum_{n=-N}^{N} \phi\{e^{-c\delta}|f(\lambda_n)|\} &\leqslant \int_{-\delta}^{\delta} dy \int_{\lambda_{-N}-\delta}^{\lambda_N+\delta} \phi\{e^{-c|y|}|f(s+iy)|\}\, ds \\
&\leqslant \int_{-\delta}^{\delta} dy \int_{-\infty}^{\infty} \phi\{e^{-c|y|}|f(s+iy)|\}\, ds \\
&\leqslant \int_{-\delta}^{\delta} dy \int_{-\infty}^{\infty} \phi\{|f(s)|\}\, ds \\
&= 2\, M\, \delta,
\end{aligned}$$

by Lemma 3 (applied to both upper and lower half-planes). Letting $N \to \infty$, we have the required result.

We can obtain a result similar to (5.2), but less precise, when the y_n are not required to be real, but satisfy for example

$$\lambda_n = \lambda_n' + i\,\lambda_n'',\ \lambda_{n+1}' - \lambda_n' \geqslant 2\,\delta > 0,\ |\lambda_n''| \leqslant \epsilon.$$

For then

$$\begin{aligned}
\pi\,\delta^2\,\phi\{e^{-c(\epsilon+\delta)}|f(\lambda_n)|\} &\leqslant \int_{\lambda_n''-\delta}^{\lambda_n''+\delta} dy \int_{\lambda_n'-\delta}^{\lambda_n'+\delta} \phi\{e^{-c(\epsilon+\delta)}|f(s+iy)|\}\, ds \\
&\leqslant \int_{-\epsilon-\delta}^{\epsilon+\delta} dy \int_{\lambda_n'-\delta}^{\lambda_n'+\delta} \phi\{e^{-c|y|}|f(s+iy)|\}\, ds
\end{aligned}$$

and so

$$\pi \delta^2 \sum_{-\infty}^{\infty} \phi \{e^{-c(\epsilon+\delta)} |f(\lambda_n)|\} \leqslant 2M(\epsilon + \delta).$$

6. For some applications it is desirable to have a result analogous to Theorem 1 when $G(z)$ is only of exponential type in a right-hand half-plane. Of course a result as precise as Theorem 1 is not to be expected; we shall consider only the question of whether

$$\int_0^{\infty} \phi\{|f(x)|\}\, dx < \infty \tag{6.1}$$

implies

$$\sum_0^{\infty} \phi\{e^{-c\delta}|f(\lambda_n)|\} < \infty. \tag{6.2}$$

This problem seems to be more difficult than the problem for the entire plane, and we may explain this qualitatively as follows. The key to Theorem 1 is Lemma 2, which extends an integrability condition from the boundary of the region (half-plane) in which $f(z)$ is defined to a line in the interior. For the corresponding result for $f(z)$ in a right-hand half-plane, we should have to extrapolate from an integrability condition on one line in the interior of the region to another. I have not been able to handle the general case, but the following theorem covers at any rate the cases of greatest interest, such as $\phi(t) = t^p$, $p > 0$, $\phi(t) = t \log^+ t$.

Lemma 4. *If $f(z)$ is of finite type in $x \geqslant 0$, if $\phi(t) = \psi(\log t)$ has the properties specified in §2, and the additional property that $\psi(-y) = O(|y|^{-1/2})$ as $y \to -\infty$, and if $\mu > 0$ and $\lambda > 2$, then*

$$\int_0^{\infty} \{\phi(|f(x)|^{\mu})\}^{\lambda}\, dx < \infty, \tag{6.3}$$

implies

$$\int_0^{\infty} \{\phi(e^{-c''\mu|y|}|f(x+iy)|^{\mu})\}^{\lambda}\, dx < \infty, \tag{6.4}$$

for each $c'' > c$ and each y.

From Lemma 4 we obtain, precisely as we obtained Theorem 1 from Lemma 3, the following result.

THEOREM 2. *If $f(z)$ is analytic and of finite type c in $x \geqslant 0$, $\phi(t)$ satisfies the hypotheses of Lemma 4, and (6.3) holds, then*

$$\sum_{n=0}^{\infty} \{\phi(e^{-c''\mu\delta}|f(\lambda_n)|^{\mu})\}^{\lambda} < \infty,$$

if $c'' > c$ and $\{\lambda_n\}$ is an increasing sequence of positive numbers with $\lambda_{n+1} - \lambda_n \geqslant 2\delta > 0$.

To prove Lemma 4, we need first the fact that $f(x)$ is bounded for real positive x (we could get along with a somewhat weaker result). This can readily be deduced from Lemma 2 along the lines of the proof of Theorem 1. It then follows by a Phragmén-Lindelöf argument that $|f(z)| \leqslant Be^{c'|y|}$, where $c' > c$ and B is a constant.

We now consider, for $x \geqslant 0$ and $y \geqslant 0$, the function

$$H(z) = |f(z)|\, e^{-c''y - \epsilon x} = |f(z)\, e^{(ic'' - \epsilon)z}|,$$

where $c'' > c'$ and $\epsilon > 0$. The function $\phi\{[H(z)]^{\mu}\}$ is subharmonic, and so by a result of Gabriel [4], if Γ is the boundary of the circular sector in the first quadrant with radii $(0, R)$ and $(0, iR)$, and C is the part of the line $y = y_0$ lying inside Γ, there is an $A(\lambda)$ depending only on λ such that

$$\int_C \{\phi([H(z)]^{\mu})\}^{\lambda}\, ds \leqslant A(\lambda) \int_{\Gamma} \{([H(z)]^{\mu})^{\lambda}\} ds$$

$$\leqslant A(\lambda) \int_0^R \{\phi[|f(x)|^{\mu}\, e^{-\epsilon\mu x}]\}^{\lambda}\, dx +$$

$$+ BA(\lambda)\, R \int_0^{\pi/2} \{\psi[\mu(c' - c'')\sin\theta - \epsilon\mu\cos\theta]\}^{\lambda}\, d\theta +$$

$$+ BA(\lambda) \int_0^R \{\psi[\mu(c' - c'')\, y]\}^{\lambda}\, dy.$$

Since $c' - c'' < 0$, $\psi(-y) = O(y^{-1/2})$ and $\lambda > 2$, the second term on the right tends to 0 as $R \to \infty$, and the third term tends to a limit L independent of ϵ. Hence

$$\int_0^{\infty} \{\phi(|f(x+iy_0)|^{\mu}\, e^{-\mu c'' y_0 - \epsilon x})\}^{\lambda}\, dx \leqslant L + A(\lambda) \int_0^{\infty} \{\phi(|f(x)|^{\mu})\, e^{-\epsilon\mu x}\}^{\lambda}\, dx.$$

Letting $\epsilon \to 0$ we have Lemma 4.

The case $\phi(t) = t^p$, $p > 1$, can be treated more simply, because by a lemma of Macintyre[3] we can write $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is an entire function of exponential type c and $f_1(x) = O(1/|x|)$ as $x \to -\infty$, while $f_2(x) = O(1/x)$ as $x \to +\infty$. Then if $f(x) \in L^p(0, \infty)$, $f_1(x) \in L^p(-\infty, \infty)$ and Plancherel and Pólya's result applies to $f_1(z)$, while $f_2(z)$ is trivial.

3. [6]. The result is not stated by Macintyre in this form; cf. [5, p. 196].

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HARTOGS' THEOREM IN COMPLEX SPACES WITH SINGULARITIES

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1. Introduction. Hartogs [3] has proved the following theorem distinctive of analytic functions in several complex variables.

THEOREM A. *If a function is defined in a domain of* E_{2k}, *and if it is analytic in each variable separately, then it is analytic in all variables.*

Note that, *ab initio*, absolutely no assumption is made on the function, as a function in all its variables jointly.

A closely related result is the following theorem [3,1]. Consider two sets of complex variables $(z_1,\ldots,z_k)$ and $(w_1,\ldots,w_l)$, and two domains D and $\tilde{D}$ in (z, w)-space, each given as a union of intersections relative to one set of variables,

$$D: \quad [z \in R, w \in H_z], \tag{1}$$

$$\tilde{D}: \quad [z \in R, w \in \tilde{H}_z], \tag{2}$$

where R is a domain in z-space, and where every connected component of $\tilde{H}_z$ contains a non-empty component of H_z. The theorem to which we are referring is the following.

THEOREM B. *If* $\phi(z,w)$ *is analytic in* (z, w) *in* D *and, for each* z *in* R, *is analytic in* w *in* $\tilde{H}_z$, *then* $\phi(z, w)$ *is analytic in* (z, w) *in* $\tilde{D}$.

This theorem obviously generalizes immediately from pairs of domains $D \subset \tilde{D}$ in the product of $E_{2k}(z)$ with $E_{2l}(w)$, to given pairs in the product of any two complex spaces $S^{2k}(z)$, $S^{2l}(w)$.

Now, in a recent paper [2] we have considered holomorphic functions on complex spaces in whose space structure certain singularities (that is, deviations from strictly Euclidean character) were allowed to occur ; and in the present paper the two theorems stated above will be carried over to products of such spaces as well.

2. Holomorphy in each variable separately. For our first result we consider two topological spaces U and V, each having the character of the space U defined in [2, Sec. 3]. We shall prove the following result.

THEOREM 1. (i) *Let $F(P,Q)$ be defined for $[P \in U, Q \in V]$. If F is holomorphic in each variable separately, then it is holomorphic in (P, Q) in (U, V).*

(ii) *The word holomorphic in* (i) *may be replaced throughout by the words strictly holomorphic.*

In the hypothesis of (i), when we say that F is holomorphic in each variable separately we mean that for each point P_0 in U, $F(P_0, Q)$ is holomorphic in V, and for each point Q_0 in V, $F(P, Q_0)$ is holomorphic in U.

Now as in [2], the space U is defined in terms of a base region R in ordinary z-space, and n polynomial relations

$$\psi_\nu(z, \zeta_1, \ldots, \zeta_\nu) = 0, \qquad \nu = 1, \ldots, n, \tag{3}$$

in $\zeta_1, \ldots, \zeta_n$, with coefficients which are analytic in R. In a similar manner, the space V is defined in terms of a base region R' in w-space, and p polynomial relations

$$\psi'_\nu(w, \xi_1, \ldots, \xi_\nu) = 0, \qquad \nu = 1, \ldots, p, \tag{4}$$

in $\xi_1, \ldots, \xi_p$ with coefficients analytic in R'. We denote by m the product $m = m_1 \ldots m_n$, where m_ν is the degree of the polynomial ψ_ν in the variable ζ_ν, and similarly we denote by m' the product $m'_1 \ldots m'_p$. As in [2] the square $\Delta^2(z)$ of the determinant of the matrix

$$\left.\begin{matrix} \left[\{[\zeta_1^{(\mu)}(z)]^{\lambda_1} \ldots [\zeta_n^{(\mu)}(z)]^{\lambda_n}\}\right] \\ \mu = 1, \ldots, m; \lambda_1 = 0, 1, \ldots, m_1 - 1; \ldots; \lambda_n = 0, \ldots, m_n - 1 \end{matrix}\right\}, \tag{5}$$

is analytic and not identically zero in R. We denote by E the subset of R where $\Delta^2(z)$ vanishes. The function $\Delta'^2(w)$ is defined in a similar manner relative to the space V.

For the proof of (i) we first introduce the functions

$$f_{\mu,\mu'}(z,w) = F(z, \zeta_1^{(\mu)}(z),\ldots,\zeta_n^{(\mu)}(z); w, \xi_1^{(\mu')}(w),\ldots, \xi_p^{(\mu')}(w)) \quad (6a)$$

and then we form the product

$$\prod_{\mu=1}^{m}\prod_{\mu'=1}^{m'}[f - f_{\mu,\mu'}(z,w)] = f^{mm'} + B_1(z,w) f^{mm'-1} + \ldots + B_{mm'}(z,w). \quad (6b)$$

Now let w^0 be any point in R'. Since $F(P,Q)$ is holomorphic in P for Q fixed in V, the symmetric functions $B_1(z,w^0),\ldots, B_{mm'}(z,w^0)$ are single-valued and analytic in z in $R-E$, and bounded in every compact set T in R. Since the set E is a removable exceptional set, see [1, p. 173], the functions $B_j(z,w^0)$, $j = 1,\ldots, mm'$, are analytic in z in R. Similarly for each point z^0 in R, the functions $B_j(z^0,w)$ are analytic in w in R'. By Hartogs' own theorem, (Theorem A), the B's are analytic in (z,w) in (R,R'). This means that they are bounded in (T,T') for every pair of compact sets

$$T \subset R, \quad T' \subset R'.$$

Hence the mm' functions (6a) are bounded in (T,T') and thus our function $F(P,Q)$ is bounded for

$$P \in U(T), \qquad Q \in V(T). \quad (7)$$

Having secured boundedness, the further reasoning is routine. We define

$$P_{\lambda_1,\ldots,\lambda_n,\lambda_1',\ldots,\lambda_n'}(z,w) = \Delta_{\lambda_1,\ldots,\lambda_n,\lambda_1',\ldots,\lambda_n'}(z,w)\,\Delta(z,w), \quad (8)$$

where $\Delta(z,w)$ is the determinant of the $(mm') \times (mm')$ matrix

$$\left[\{[\zeta_1^{(\mu)}(z)]^{\lambda_1}\cdots[\zeta_n^{(\mu)}(z)]^{\lambda_n}[\xi_1^{(\mu')}(w)]^{\lambda_1'}\cdots[\xi_p^{(\mu')}(w)]^{\lambda_p'}\}\right], \quad (9)$$

for

$$\lambda_1 = 0, 1,\ldots, m_1 - 1;\ldots;\lambda_n = 0, 1,\ldots; m_n - 1,$$
$$\lambda_1' = 0, 1,\ldots, m_1' - 1;\ldots;\lambda_p' = 0, 1,\ldots; m_p' - 1, \quad (10)$$

and

$$\mu = 1, \ldots, m, \qquad \mu' = 1, \ldots, m', \tag{11}$$

and where $\Delta_{\lambda_1, \ldots, \lambda_n, \lambda_1', \ldots, \lambda_n'}(z, w)$ is the determinant of the matrix obtained from the matrix (9) by replacing the elements of the column, one element of which is displayed in (9), by $f_{\mu,\mu'}(z, w)$ for μ, μ' ranging over (11). By symmetry and holomorphy of F in each variable separately we conclude as in the case of the B's that each of the functions (8) is analytic in (z, w) in (R, R').

By (6a) and (8) we see that

$$\Delta^2(z, w)\, F(z, \zeta; w, \xi) = \sum_{\lambda, \lambda'} P_{\lambda_1, \ldots, \lambda_n, \lambda_1', \ldots, \lambda_n'}(z, w)\, \zeta_1^{\lambda_1} \ldots \zeta_n^{\lambda_n}\, \xi_1^{\lambda'_1} \ldots \xi_p^{\lambda'_p} \tag{12}$$

in (U, V) where the sum is taken over (10).

We observe that $\Delta^2(z, w)$ is analytic and not identically zero in (R, R'), being given as the product

$$\Delta^2(z, w) = [\Delta(z)]^{2m'}\, [\Delta'(w)]^{2m}. \tag{13}$$

Thus the relation (12), together with the boundedness of F in (7), yields the desired conclusion of (i).

3. **Strict holomorphy in each variable.** For the proof of (ii) we will utilise the boundedness of F in $U(T)$, $V(T')$ and the relation (12) already obtained. By the strict holomorphy of $F(P, Q)$ in Q for P fixed we also have

$$F(z, \zeta; w, \xi) = \sum_{\lambda'} A_{\lambda'}(z, \zeta; w)\, \xi_1^{\lambda'_1} \ldots \xi_p^{\lambda'_p}, \tag{14}$$

in (U, V), where the A's are analytic in w in R' for each fixed point (z, ζ) in U. The relation (14) and the boundedness of F in (7) tells us that the functions

$$\Delta'^2(w)\, A_{\lambda'}(z, \zeta; w) \tag{15}$$

are bounded for

$$[(z, \zeta) \in U(T),\ w \in T']. \tag{16}$$

Since $\Delta'^2(w)$ and the A's are analytic in w, by Lemma 4.4 of [2] we conclude that the functions

$$A_{\lambda'}(z, \zeta; w) \tag{17}$$

themselves are bounded in (16) even though $\Delta'(w)$ may have zeros in R'. We will use this fact a little later.

Next, a comparison of (12) and (14) gives

$$\Delta^2(z, w)\, A_{\lambda'}(z, \zeta; w) = \sum_{\lambda} P_{\lambda, \lambda'}(z, w)\, \zeta_1^{\lambda_1} \dots \zeta_n^{\lambda_n}, \tag{18}$$

for $[(z, \zeta) \in U, w \in R']$. By solving for $P_{\lambda, \lambda'}$, we now obtain

$$\Delta^2(z)\, P_{\lambda, \lambda'}(z, w) = \Delta^2(z, w)\, A_{\lambda, \lambda'}(z, w)\, \Delta(z), \tag{19}$$

where $\Delta(z)$ is the determinant of the $m \times m$ matrix (5) and $A_{\lambda, \lambda'}(z, w)$ is the determinant of the matrix obtained from (5) by replacing the elements of the column, one element of which is displayed in (5), by

$$A_{\lambda'}[z, \zeta_1^{(\mu)}(z), \dots, \zeta_n^{(\mu)}(z); w], \tag{20}$$

for $\mu = 1, \dots, m$. By symmetry, boundedness of (17) in (16), and Hartogs' theorem, we conclude that the functions

$$Q_{\lambda, \lambda'}(z, w) = A_{\lambda, \lambda'}(z, w)\, \Delta(z), \tag{21}$$

are analytic in (z, w) in (R, R'). Thus by (19)

$$\Delta^2(z)\, P_{\lambda, \lambda'}(z, w) = \Delta^2(z, w)\, Q_{\lambda, \lambda'}(z, w), \tag{22}$$

where $Q_{\lambda, \lambda'}(z, w)$ is analytic in (R, R'). This means that we can replace the representation (12) by

$$\Delta^2(z)\, F(z, \zeta; w, \xi) = \sum_{\lambda, \lambda'} Q_{\lambda, \lambda'}(z, w)\, \zeta_1^{\lambda_1} \dots \zeta_n^{\lambda_n}\, \xi_1^{\lambda'_1} \dots \xi_p^{\lambda'_p}. \tag{23}$$

Similarly, on interchanging the roles of z and w, we obtain a representation (23′) with $\Delta^2(z)$ replaced by $\Delta'^2(w)$ and $Q_{\lambda, \lambda'}(z, w)$ by $R_{\lambda, \lambda'}(z, w)$, where the R's are analytic in (R, R'). On comparing (23) and (23′) we obtain the equalities

$$\Delta'^2(w)\, Q_{\lambda, \lambda'}(z, w) = \Delta^2(z)\, R_{\lambda, \lambda'}(z, w), \tag{24}$$

throughout (R, R').

We now use the following known lemma on analytic functions of several variables, from the local theory of ideals.

Lemma A. *If $\Phi(z, w)$ and $\Psi(z, w)$ are analytic for $z \in R$, $w \in R'$, if $\phi(w)$ is analytic and $\not\equiv 0$ in R, if $\psi(z)$ is analytic and $\not\equiv 0$ in R', and if*

$$\phi(w)\, \Phi(z, w) = \psi(z)\, \Psi(z, w) \qquad \textit{in } (R, R'),$$

then there is an analytic function $\Psi_1(z, w)$ *in* (R, R') *such that*

$$\Phi(z, w) = \psi(z)\, \Psi_1(z, w), \qquad \textit{in } (R, R').$$

In other words, if $\psi(z)$ *is a factor of* $\phi(w)\, \Phi(z, w)$, *then it is a factor of* $\Phi(z, w)$ *itself.*

By this lemma, relation (24) tells us that there are functions

$$C_{\lambda, \lambda'}(z, w)$$

analytic in (R,R') such that

$$Q_{\lambda, \lambda'}(z, w) = \Delta^2(z)\, C_{\lambda, \lambda'}(z, w).$$

Thus we may replace the representation (23) by

$$F(z, \zeta; w, \xi) = \sum_{\lambda, \lambda'} C_{\lambda, \lambda'}(z, w)\, \zeta_1^{\lambda_1} \ldots \zeta_n^{\lambda_n}\, \xi_1^{\lambda_1'} \ldots \zeta_p^{\lambda_p'}. \tag{25}$$

This means that F is strictly holomorphic in (R, R'), and thus the proof of part (ii) of Theorem 1 is complete.

4. Holomorphic continuation. We now consider two domains (1) and (2) in (z, w)-space. Let R' be a domain in w-space which contains the union of the sets $\tilde{H}_z$,

$$\sum_{z \in R} \tilde{H}_z \subset R', \tag{26}$$

and as in Section 2 consider the two spaces U and V defined in terms of the polynomial relations (3) and (4) respectively for z in R and w in R'.

For z fixed in R we denote by V_z the subset of V for which $w \in H_z$, and by $\tilde{V}_z$ the subset of V for which $w \in H_z$. If P is any point (z, ζ) in U, we will also denote V_z by V_P and $\tilde{V}_z$ by $\tilde{V}_P$.

We shall prove the following theorem.

THEOREM 2. (i) *If* $F(P, Q)$ *is holomorphic in* (P, Q) *in*

$$[P \in U,\ Q \in V_P], \tag{27}$$

and for each P *in* U *is holomorphic in* Q *in* $\tilde{V}_P$, *then* F *is holomorphic in* (P, Q) *in*

$$[P \in U,\ Q \in \tilde{V}_P]. \tag{28}$$

(ii) *The word holomorphic in* (i) *may be replaced throughout by the words strictly holomorphic,*

We will first prove part (ii), that is the strictly holomorphic version. By hypothesis we have a representation (25) in (27), where the C's are analytic in (1), and we have a representation (14) in (28), where the A's are analytic in w in $\tilde{H}_z$ for each fixed point (z, ζ) in U. A comparison of these two representations yields the relations

$$A_{\lambda'}(z, \zeta; w) = \sum_{\lambda} C_{\lambda,\lambda'}(z, w)\, \zeta_1^{\lambda_1} \ldots \zeta_n^{\lambda_n}, \tag{29}$$

valid for $[(z, \zeta) \in U,\ w \in H_z]$. We may solve for the C's in terms of the A's, obtaining

$$\Delta^2(z)\, C_{\lambda,\lambda'}(z, w) = A_{\lambda,\lambda'}(z, w)\, \Delta(z) = Q_{\lambda,\lambda'}(z, w), \tag{30}$$

where the $A_{\lambda,\lambda'}$ and $Q_{\lambda,\lambda'}$ are defined as in (19) and (21).

Now the left-hand side of (30) is analytic in (z, w) in (1), while the right hand side is analytic in w in $\tilde{H}_z$ for each fixed z in R. Thus by Theorem B, the function $Q_{\lambda,\lambda'}(z, w)$ is analytic in (z, w) in (2). The desired result on the strict holomorphy of F in (28) will follow from a lemma on analytic functions which we next state and prove.

LEMMA 1. *If $C(z, w)$ is analytic in* (1), *if $Q(z, w)$ is analytic in* (2), *if $\phi(z)$ is analytic and not identically zero in R and if*

$$\phi(z)\, C(z, w) = Q(z, w)$$

holds in (1), *then $C(z, w)$ has an analytic continuation into* (2).

For the proof it is sufficient to establish the conclusion when H_z and $\tilde{H}_z$ are multi-cylinders independent of z,

$$H_z : |w_j| < p_j;\ \tilde{H}_z : |w_j| < P_j,\ j = 1, \ldots, l,$$

where $0 < p_j < P_j$.

We develop $C(z, w)$ in a series

$$C(z, w) = \sum_{n_1,\ldots,n_l=0}^{\infty} c_{n_1,\ldots,n_l}(z) w_1^{n_1} \ldots w_l^{n_l}. \tag{31}$$

The coefficients $c_{n_1,\ldots,n_l}(z)$ are analytic in R and the series (31) is absolutely-continuously convergent in $[z \in R, |w_j| < p_j]$. On the other hand the series

$$\sum \phi(z).\ c_{n_1,\ldots,n_l}(z) w_1^{n_1} \ldots w_l^{n_l} \tag{32}$$

is absolutely-continuously convergent in $[z \in R, |w_j| < P_j]$. From this latter fact, if T is any compact set in R and if $0 < P'_j < P_j$, we see that there is a positive number $M = M(T, P'_1, \dots, P'_l)$ such that

$$|\phi(z)c_{n_1,\dots,n_l}(z)P'^{n_1}_1 \dots P'^{n}| \leqslant M, \text{ for } z \in T, n_1, \dots, n_l = 0, 1, \dots. \quad (33)$$

Since ϕ and the c's are analytic in R, we conclude from (33) and Lemma 4.4 of [2] that there is a constant M' such that

$$c_{n_1,\dots,n_l}(z)P'^{n_1}_1 \dots P'^{n_l}_l| \leqslant M', \text{ for } z \in T, n_1, \dots, n_l = 0, 1, 2, \dots. \quad (34)$$

Thus the series (31) converges absolutely-continuously in $[z \in R, |w_j| < P'_j]$ and since P'_j is any number less than P_j, the same is true if $[z \in R, |w_j| < P_j]$. This concludes the proof of the lemma.

Applying the lemma to the functions C in (30) we see that the $C_{\lambda,\lambda'}$ are analytic in (2) and hence F is strictly holomorphic in (28). This yields part (ii) of Theorem 2.

For part (i) of the theorem, we first note that since F is holomorphic in (27) the function

$$\Delta^2(z, w)\; F(z, \zeta; w, \xi) \quad (35)$$

is strictly holomorphic there. On the other hand, from the holomorphy of $F(P, Q)$ for Q in V_p for P fixed in U we know that

$$\Delta'^2(w)\; F(z, \zeta; w, \xi)$$

is holomorphic for (w, ξ) in $\tilde{V}_z$ for (z, ζ) fixed in U. By the relation (13) this latter fact tells us that (35) is strictly holomorphic for (w, ξ) in $\tilde{V}_z$ for (z, ζ) fixed in U. Hence part (ii) of Theorem 2 applies to the function (35), and we conclude that (35) is strictly holomorphic in (28). If in addition we can show that F itself is bounded in every compact subset of (28) then we will have F holomorphic in (28).

In order to prove boundedness of F we define the functions (6a) and the symmetric functions $B_1(z, w), \dots, B_{mm'}(z, w)$ in (6b). By the holomorphy of F in (P, Q) in (27) the B's are analytic in (z, w) in (1), and by the holomorphy of F in Q in $\tilde{V}_P$

for P fixed in U, the B's are analytic in w in $\tilde{H}_z$ for z fixed in R. Hence by Theorem B they are analytic in (z, w) in (2). Thus $F(z, \zeta; w, \xi)$ is bounded whenever (z, w) belongs to any compact subset of (2). This, together with the strict holomorphy of (35) in (28) yields the holomorphy of F in (28).

This concludes the proof of Theorem 2.

We next give a global version of Theorem 2.

First we introduce the concept of a *full-part* of a space V of the character considered in Section 2. A subset W of V is called a *full-part* of V if it consists of all points $[w, \xi_1^{(\mu)}(w), \ldots, \xi_p^{(\mu)}(w)]$ for w in an open set of the base region R' and $\mu = 1, \ldots, m'$.

Next, let $S_{2k}^{(w)}$ be a complex space with singularities, defined as in [2, Sec. 10], with a covering by open sets, $\{V_\sigma\}$, each having the character of the space V. We say that $S_{2k}^{(w)}$ is *fully-connected* if to every V_0 in $\{V_\sigma\}$ and every point Q in $S_{2k}^{(w)}$ there exists a finite chain, $V_1, \ldots, V_r$, with $Q \in V_r$, such that the intersection $V_{\tau+1} \cap V_\tau$ contains a full-part of $V_{\tau+1}$, $\tau = 1, \ldots, r-1$.

As a corollary to Theorem 2 we have the following result.

COROLLARY : (i) *Let $S_{2k}^{(w)}$ be a fully-connected complex space with singularities, and let V_0 be any element of the covering $\{V_\sigma\}$. If $F(P, Q)$ is holomorphic in (P, Q) in*

$$P \in U, \; Q \in V_0$$

and holomorphic in Q in $S_{2k}^{(w)}$ for each fixed point P in U, then F is holomorphic in (P, Q) in

$$P \in U, \; Q \in S_{2k}^{(w)}.$$

(ii) *In* (i) *the word holomorphic may be replaced throughout by the words strictly holomorphic.*

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ON THE DISTRIBUTION OF THE ZEROS OF AN ENTIRE FUNCTION

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The following brief note was suggested by the problem of relating the order ρ and type τ of an entire function $f(z)$ to the distribution function $n(x) = \{$number of zeros of f in $|z| \leqslant x\}$. It is known that [Titchmarsh, 2, p. 249] :

$$\limsup x^{-\rho} \int_0^x n(t)\, t^{-1}\, dt \leqslant \tau. \qquad (1)$$

Denoting the upper and lower limits of $n(x)\, x^{-\rho}$ by L and l we also have [Boas, 1] :

$$l \leqslant \rho\,\tau \text{ and } L \leqslant e\,\rho\,\tau. \qquad (2)$$

These are best possible; in connection with the second, an easier example than that used by Boas can be given. Choose $f(z) = \prod_1^\infty [1 - (z\,R^{-n^2})^{2n^2}]$ where $R > 1$; this is of order $\rho = (\log 2)/\log R$, and $L = 1$. Writing $f(z) = \Sigma A_m z^m$, $e\,\rho\,\tau = \limsup\ m\,|A_m|^{\rho/m}$. If $m = \Sigma\, 2^{n_j^2}$ and $m^* = \Sigma n_j^2\, 2^{n_j^2}$, then $|A_m|^{\rho/m} = 2^{-m^*/m}$ and taking $m = 1 + 2 + 2^4 + \dots + 2^{n^2}$ for $n = 1, 2, \dots$, we find $e\,\rho\,\tau = 1$.

The point of this note is the observation that it is not possible to have equality in both cases in (2) at the same time. Let Ω be the set of points (l, L) which correspond to possible functions $f(z)$ for a fixed choice of $\rho > 0$ and $\tau > 0$; it will be evident that we may assume $\rho\,\tau = 1$. Ω then lies in the rectangle with vertices (0,0), (1,0), (0,e), (1,e) by (2). We show that the upper right corner may be deleted by a cut through (0,e). We phrase our main result as a tauberian-type theorem.

Let $\phi(x)$ be positive, continuous, and for some $c > 0$, $\phi(x) \sim c/x$. Let

$$g(x) = \int_{0+}^{x} f(t)\,\phi(t)\,dt. \tag{3}$$

Put $\psi(x) = \exp \int_0^x \phi$, and let A, a be the upper and lower limits of $f(x)/\psi(x)$, and B, b be those of $g(x)/\psi(x)$. It follows at once that $a \leqslant b \leqslant B \leqslant A$ so that in particular, if $f(x) \sim \psi(x)$, then $g(x) \sim \psi(x)$. In general, the converse of this is false. Under the special assumption that $f(x)$ is increasing, we wish to estimate A and a in terms of B.

THEOREM. *If $f(x)$ is increasing, and $B \leqslant 1$, then*

$$0 \leqslant a + e\,A \leqslant e^2. \tag{4}$$

Choose $\lambda < 1$ and $\mu < 1$. Then, by (3)

$$\begin{aligned} g(x) &\geqslant \int_{\lambda x}^{x} f\phi + \int_{\mu\lambda x}^{\lambda x} f\phi \\ &\geqslant f(\lambda x) \int_{\lambda x}^{x} \phi + f(\mu\lambda x) \int_{\mu\lambda x}^{\lambda x} \phi \\ &\geqslant f(\lambda x) \log\,[\psi(x)/\psi(\lambda x)] + f(\mu\lambda x) \log\,[\psi(\lambda x)/\psi(\mu\lambda x)], \end{aligned}$$

so that

$$\frac{g(x)}{\psi(x)} \geqslant \frac{f(\lambda x)}{\psi(\lambda x)} \frac{\psi(\lambda x)}{\psi(x)} \log \frac{\psi(x)}{\psi(\lambda x)} + \frac{f(\mu\lambda x)}{\psi(\mu\lambda x)} \frac{\psi(\mu\lambda x)}{\psi(\lambda x)} \frac{\psi(\lambda x)}{\psi(x)} \log \frac{\psi(\lambda x)}{\psi(\mu\lambda x)}.$$

From our assumption on $\phi(x)$, $\lim \psi(sx)/\psi(x) = s^c$ for all s. Hence,

$$1 \geqslant B \geqslant A\lambda^c\, c \log\,(1/\lambda) + a\lambda^c\, \mu^c\, c \log\,(1/\mu).$$

In this, put $\mu = e^{-1/c}$ and $\lambda = e^{-t/c}$, and obtain

$$a + e\,t\,A \leqslant e^{t+1}, \tag{5}$$

which holds for all $t > 0$. The choice $t = 1$ gives (4).

This can be improved; since (5) holds for all t, we may compute the envelope of the family of halfspaces, and get a sharper relation connecting A and a,

$$a \leqslant e\,A \log (e/A). \tag{6}$$

Since we have $0 \leqslant a \leqslant 1$, (6) is only slightly better than (4).

To apply this to the case in question, take $\phi(x) = \rho/x$ and $f(x) = n(x)$; from (1), $B \leqslant \rho\tau$, and applying the theorem:

Corollary 1. $l + e\,L \leqslant e^2\rho\tau$.

This in turn shows that equality in (2) cannot hold.

Corollary 2. *If* $L = e\rho\tau$, *then* $l = 0$. *Likewise, if* $l = \rho\tau$, *then* $L \leqslant (e - e^{-1})\,\rho\tau = (2.35\ \ldots)\,\rho\tau$.

If the sharper relation (6) is used, we find that if $l = \rho\tau$, $L \leqslant (2.32)\,\rho\tau$. This shows how slight a gain (6) is over (4). It seems dubious that these relations are best possible.

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ERRATA

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p. 12, lines 10, 14, read "konvexen" for "konvexer"

p. 12, line 14, read "Ausfüllung" for "Aüsfullung"

p. 14, line 7, and p. 24, line 9, read "McShane" for "Mcshane"

p. 25, line 27, read "Theorem" for "Therem"

p. 29, line 8, read "polynomial" for "polyinomial"

p. 29, line 11, read "*holomorphic*" for "*holmorphic*"

p. 41, line 18, delete "$-\frac{1}{2}x^2$"

p. 41, line 19, insert "; $-\frac{1}{2}x^2$" after "$a_r+\nu-\rho_n+1$"

p. 43, line 11, read "${}_nF_{n-1}[a_r,\ldots$" for "${}_nF_{n-1}[a',\ldots$"

p. 44, line 16, read "$\frac{1}{2}(a_1+m_{n-1}+1)$;" for "$\frac{1}{2}(a_1+m_{n-1}+1$;"

p. 44, line 7 up, "$-\frac{1}{2}t^2$" misplaced

p. 44, line 2 up, and line 5 up, delete "$-\frac{1}{2}x^2$"

p. 44, last line, and line 4 up, insert "; $-\frac{1}{2}x^2$" as the last term inside the bracket

p. 45, line 8, read "(7) 25 (1938)" for "(1938), §§ 7-25"

p. 135, line 11 up, read "NIKOLSKI" for "NIKOLSKII"

p. 176, line 3 up, read "*Trans.*" for "*Tras.*"

p. 191, line 5 up, read "$-\frac{\partial\chi'}{\partial y}$" for "$-\frac{\partial\chi'}{\partial x}$"

p. 192, line 10 up, read "$r=1,2.$" for "$r=1,2,\ldots.$"

p. 194, lines 3, 8, read "$F(C)$" for "$F(c)$"

p. 194, line 9 up, read "$\bar{\psi}$" for "ψ"

p. 195, line 6, read "γ" for "ν"

p. 195, lines 14, 16, read "c" for "C"

p. 196, line 3, read "superposability" for "superposibility"

p. 196, line 9, read "$e^{i(\alpha_1x+\alpha_2y)}$" for "$e^{i(\alpha_1\alpha+z_2y)}$"

p. 197, line 6 up, read "RATIB" for "RÁTIB"

... JAN 1953

HALF-BOUNDED MATRICES

J. S. MacNERNEY

[Received August 2, 1952]

The real part, imaginary part, modulus, and conjugate of a complex number z are denoted by Re z, Im z, $|z|$, and z^*, respectively. By a *matrix* A we mean an infinite complex matrix (A_{pq}) with indices ranging over the positive integers, and we denote its conjugate transpose (or adjoint) by A^*. For each positive integer n, the *n-th segment* of the matrix A is the n-by-n matrix B such that $B_{pq}=A_{pq}$ for $p, q=1,\dots,n$. The identity matrix is denoted by I (i.e., I_{pq} denotes Kronecker's delta), and the matrix B is a *reciprocal* of the matrix A provided that $AB=I$ and $BA=I$. S is the linear space of all sequences $x=(x_1,x_2,\dots)$ of complex numbers, with the following definitions: $\theta=(0,0,\dots)$, and if c is a complex number and x, y is a sensed pair of sequences in S, then

$$x+y=(x_1+y_1, x_2+y_2,\dots), \qquad cx=(cx_1, cx_2,\dots),$$

and for each positive integer n and each matrix A,

$$Ax=(\Sigma A_{1r}x_r, \Sigma A_{2r}x_r,\dots),$$

$$A_nx=(\sum_{r=1}^{n} A_{1r}x_r, \sum_{r=1}^{n} A_{2r}x_r,\dots),$$

$$(x,y)_n=\sum_{r=1}^{n} x_ry^*_r, \qquad \|x\|_n=(x,x)_n^{1/2}, \text{ and}$$

$$(x,y)=\lim\,(x,y)_n \text{ as } n\to\infty.$$

S_0 denotes the Hilbert space which is that subspace of S consisting of all sequences x in S such that (x,x) exists.

1. Introduction. The *numerical range*, $W(A)$, of a matrix A is defined as the set of all complex numbers w such that for some positive integer n and some sequence x in S

$$w=(A_nx, x)_n \text{ and } (x,x)_n=1; \tag{1.1}$$

Hausdorff [2] proved that the numerical range of an n-by-n matrix is a closed convex set—hence $W(A)$ is convex. A *half-bounded* matrix is a matrix M such that $W(M)$ is in the exterior of some half-plane (i.e. lies entirely on one side of some line in the complex plane) : since the numerical range of a matrix is a convex set, half-bounded matrices are those matrices M such that $W(M)$ is not the entire plane. Since Hermitian matrices are those matrices H such that $W(H)$ is real (i.e. $H = H^*$), and bounded matrices are those matrices B such that $W(B)$ is bounded, this following from the familiar decomposition

$$4\,(B_n x, y)_n = (B_n[x+y], x+y)_n - (B_n[x-y], x-y)_n + \\ + i(B_n[x+iy], x+iy)_n - i(B_n[x-iy], x-iy)_n, \quad (1.2)$$

we see that the class of half-bounded matrices includes all Hermitian matrices and all bounded matrices.

The matrix A is *squarable* provided A^*A exists, and is *completely squarable* provided A^*A exists and AA^* exists. We see that a necessary and sufficient condition that the matrix A should be squarable is that, for each sequence x in S and each positive integer n, the sequence $A_n x$ should be in S_0. The reader is referred to von Neumann ([5], Appendix III) concerning the relationship of squarable matrices and of completely squarable matrices with closed linear operators in abstract Hilbert space.

This paper is concerned primarily with the existence and nature of reciprocals of matrices of the form $zI-A$, where z is a complex number and A is a completely squarable, half-bounded matrix. Our principal results are summarized as follows :

In section 2, we suppose that A is a completely squarable, half-bounded matrix and prove that there is a matrix $B(z)$ of functions, regular for z not in the closure of $W(A)$, which is a reciprocal of $zI-A$ throughout this domain, which is bounded by the reciprocal of the distance from z to $W(A)$, and which has

its numerical range determined by supporting lines to the set $W(A)$. It is shown that an arbitrary reciprocal of $zI-A$, for z ranging over some unbounded set, determines A in various asymptotic senses depending on types of boundedness assumed for it. These ideas are extended to reciprocals of $zI-A-E(z)$, where $E(z)$ is a half-bounded matrix of regular functions.

In section 3, we give a theorem about Stieltjes transforms of bounded non-decreasing functions, extend this general idea to Stieltjes transforms of *non-decreasing matrices* (Definition 3.1), and obtain necessary and sufficient conditions that a matrix of functions should be the Stieltjes transform of a non-decreasing matrix. In this way we characterize certain of the reciprocals found in section 2. We establish the analogue of Hamburger's theorem ([1], Theorem IX) on the existence of " moment " integrals, an integral representation for squarable Hermitian matrices, and a " continued fraction " algorithm for Stieltjes transforms of unit non-decreasing matrices where the first two moment integrals are bounded.

In section 4, we apply some of the foregoing ideas to the theory of positive definite continued fractions and prove, for example, that the equivalent functions (see Wall [7], p. 231) of a positive definite J-fraction are exactly those functions giving rise to Stieltjes transforms of unit non-decreasing matrices as reciprocals of the associated J-forms.

2. Existence of reciprocals.

Suppose A is a half-bounded matrix. Then there exist a complex number u with modulus 1 and a real number v such that

$$\mathrm{Re}[u^*(A_n x, x)_n] \leqslant v(x,x)_n \text{ for } n = 1,2,\ldots, \text{ all } x \text{ in } S. \quad (2.1)$$

Let U be the set of all such u, and for each u in U let $v(u)$ be the least number v such that (2.1) holds. We see that the closure of $W(A)$ is the set of all complex numbers w such that

$$\mathrm{Re}(u^* w) \leqslant v(u) \text{ for each } u \text{ in } U. \quad (2.2)$$

For each complex z not in the closure of $W(A)$, let $d(z)$ denote the distance from z to the boundary of $W(A)$.

THEOREM 2.1. *There is a matrix $B(z)$ of functions such that the following conditions hold:*

(i) *$B(z)_{pq}$ is regular for $d(z) > 0$,* $(p, q = 1, 2, \dots)$,

(ii) *for each u in U and each z such that* $\text{Re}(u^*z) > v(u)$ *and each positive integer n and each sequence x in S,*

$$[\text{Re}(u^*z) - v(u)] \| B(z)_n x \|_n^2 \leqslant \text{Re}[u(B(z)_n x, x)_n],$$

(iii) *if A is symmetric, then $B(z)$ is symmetric for $d(z) > 0$,*

(iv) *if A is completely squarable, then, for each z not in the closure of $W(A)$, $B(z)$ is a reciprocal of $zI - A$.*

PROOF. For each u in U and each complex z and each positive integer n and each x in S

$$\text{Re}[u^*(zx - A_n x, x)_n] \geqslant [\text{Re}(u^*z) - v(u)] \| x \|_n^2; \qquad (2.3)$$

hence the determinant of the n-th segment of $zI - A$ is not zero for $d(z) > 0$, and there is a matrix $B^n(z)$ such that the n-th segment of $B^n(z)$ is the reciprocal of the n-th segment of $zI - A$ and $B^n(z)_{pq} = 0$ if either $p > n$ or $q > n$. We note that if A is symmetric (i.e. $A_{pq} = A_{qp}$ for $p, q = 1, 2, \dots$) then, for each positive integer n, $B^n(z)$ is symmetric. From (2.3) we now have, for each u in U and each z such that $\text{Re}(u^*z) > v(u)$ and each positive integer m and each x in S,

$$\text{Re}[u^*(x, B^m(z)_m x)_m] \geqslant [\text{Re}(u^*z) - v(u)] \| B^m(z)_m x \|_m^2, \qquad (2.4)$$

from which it follows that, for $n \leqslant m$,

$$[\text{Re}(u^*z) - v(u)] \| B^m(z)_n x \|_n^2 \leqslant \text{Re}[u(B^m(z)_n x, x)_n]. \qquad (2.5)$$

Now (2.5) is equivalent (for the indicated values of u, z, m, and n) to the relationship

$$\| B^m(z)_n x - \frac{u^*}{2\,\text{Re}[u^*z - v(u)]} x \|_n \leqslant \frac{1}{2\,\text{Re}[u^*z - v(u)]} \| x \|_n, \qquad (2.6)$$

for all x in S, and this in turn implies that for each x and y in S

$$|(B^m(z)_n x, y)_n| \leqslant \frac{1}{\text{Re}[u^*z - v(u)]} \| x \|_n \| y \|_n. \tag{2.7}$$

From (2.7) we see that if b is a positive number, then for each positive integer m and each z such that $d(z) \geqslant 1/b$

$$\sum_{(r)} | B^m(z)_{pr} |^2 \leqslant b^2 \text{ for } p = 1, 2, \ldots, \tag{2.8}$$

$$\sum_{(r)} | B^m(z)_{rq} |^2 \leqslant b^2 \text{ for } q = 1, 2, \ldots, \tag{2.9}$$

and, in particular,

$$| B^m(z)_{pq} | \leqslant b \text{ for } p, q = 1, 2, \ldots. \tag{2.10}$$

Now, for each sensed triple m, p, q, of positive integers, $B^m(z)_{pq}$ is a rational function : moreover, we see that the set of all z such that $d(z) > 0$ is either a connected domain or the sum of two open half-planes. We now employ a form of Stieltjes' convergence continuation process (see, for example, Wall [7], p. 105, Theorem 24.1). It follows from (2.10) that there exist a matrix $B(z)$ of functions, regular for $d(z) > 0$, and an unbounded set M of positive integers such that, for $p, q = 1, 2, \ldots,$

$$B^m(z)_{pq} \to B(z)_{pq}, \text{ as } m \to \infty \text{ in } M, \tag{2.11}$$

uniformly over each closed bounded set of z's such that $d(z) > 0$, and $B(z)$ is symmetric if A is symmetric. Of course the set M, independent of p and q, may be thought of as being obtained by applying a diagonal process to a collection of simple sequences of functions, such as

$$B^m_{11}, B^m_{12}, B^m_{21}, B^m_{13}, B^m_{22}, B^m_{31}, \ldots \qquad (m = 1, 2, \ldots):$$

we omit the details. From (2.5) we see that part (ii) of Theorem 2.1 holds. Now, for each sensed pair p,q of positive integers and each z such that $d(z) > 0$,

$$\sum_{(r)} B^m(z)_{pr}[zI - A]_{rq} = \sum_{r=1}^{m} B^m(z)_{pr}[zI - A]_{rq}, \tag{2.12}$$

which is I_{pq} for $m \geqslant q$, and

$$\sum_{(r)} [zI - A]_{pr} B^m(z)_{rq} = \sum_{r=1}^{m} [zI - A]_{pr} B^m(z)_{rq}, \tag{2.13}$$

which is I_{pq} for $m \geqslant p$. If we suppose A is completely squarable then, by (2.8) and (2.9) and the well-known continuity of bounded linear forms under weak convergence in S_0, we have part (iv), and Theorem 2.1 is established.

We note that if $B(z)$ is a matrix of functions satisfying condition (ii) of Theorem 2.1 then the relations (2.7), (2.8) and (2.9) hold with $B^m(z)$ replaced by $B(z)$. If A is completely squarable and part (iv) of Theorem 2.1 holds for such a matrix $B(z)$, then $B(z)$ determines A in certain asymtotic senses—as is seen from the following sequence of three theorems. That this asymptotic determination of A is independent of the assumed half-boundedness when A is squarable is shown by proceeding as follows:

We suppose that C is a squarable matrix and $D(z)$ is a matrix of complex valued functions defined over the unbounded set Z in the plane such that

$$D(z)\,[zI - C] = I \text{ for each } z \text{ in } Z. \tag{2.14}$$

THEOREM 2.2. *If p is a positive integer and b is a positive number such that $\Sigma\,|D(z)_{pr}|^2 \leqslant b^2$ for each z in Z, then for each positive integer q*

$$zD(z)_{pq} \to I_{pq} \text{ as } z \to \infty \text{ in } Z. \tag{2.15}$$

PROOF. Suppose p is a positive integer and b is a positive number such that $\Sigma\,|D(z)_{pr}|^2 \leqslant b^2$ for each z in Z. Then from Schwarz's inequality and (2.14) we have for each positive integer q

$$|zD(z)_{pq} - I_{pq}|^2 \leqslant (b^2)\,\Sigma\,|C_{rq}|^2 \text{ for each } z \text{ in } Z. \tag{2.16}$$

Hence, for each positive integer q, $D(z)_{pq} \to 0$ as $z \to \infty$ in Z and, by the continuity of bounded linear forms mentioned previously, we obtain (2.15).

THEOREM 2.3. *If p is a positive integer and b is a positive number such that $\Sigma\,|zD(z)_{pr}|^2 \leqslant b^2$ for each z in Z, then for each positive integer q*

$$z[zD(z)_{pq} - I_{pq}] \to C_{pq} \text{ as } z \to \infty \text{ in } Z. \tag{2.17}$$

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PROOF. Suppose p is a positive integer and b is a positive number such that $\Sigma|zD(z)_{pr}|^2 \leqslant b^2$ for each z in Z. Then from Schwarz's inequality and (2.14) we have for each positive integer q

$$|z[zD(z)_{pq} - I_{pq}]|^2 \leqslant (b^2)\,\Sigma|C_{rq}|^2 \text{ for each } z \text{ in } Z. \tag{2.18}$$

Hence, for each positive integer q, (2.15) holds and, again by the continuity of bounded linear forms, we obtain (2.17) — and the theorem is established.

THEOREM 2.4. *If C is completely squarable and $D(z)$ is a reciprocal of $zI - C$ for each z in Z and b is a positive number such that for each positive integer n and each x and y in S*

$$|z(D(z)_n x, y)_n| \leqslant b\,\|x\|_n\,\|y\|_n \text{ for each } z \text{ in } Z, \tag{2.19}$$

then for all such n, x, y, and z the modulus of

$$z^3(D(z)_n x, y)_n - z^2(x, y)_n - z(C_n x, y)_n \tag{2.20}$$

is not greater than $b\,\|C_n x\|\;\|C_n^* y\|$, so that (2.17) holds for each sensed pair p, q of positive integers.

PROOF. Suppose C is completely squarable and $D(z)$ is a reciprocal of $zI—C$ for each z in Z and b is a positive number such that (2.19) holds for each positive integer n and each x and y in S. Then $zD(z)$ is bounded with norm not greater than b for all z in Z. Hence we have, for each z in Z, the matric equation

$$z^3 D(z) - z^2 I - zC = C[zD(z)]\,C, \tag{2.21}$$

where the product on the right is associative. Applying (2.19) to the appropriate bilinear forms from (2.21), we readily obtain the required bound for (2.20)—this completes the proof.

The following theorem concerning the case when the matrix C is "column-finite" is of some interest in the theory of J-matrices :

THEOREM 2.5. *If $(n_1, n_2, \ldots)$ is a sequence of positive integers such that $C_{pq} = 0$ for $p \geqslant n_q$ $(q = 1, 2, \ldots)$ and E is a matrix of positive numbers such that $|D(z)_{pq}| \leqslant E_{pq}$ for each z in Z*

$(p, q = 1, 2, \ldots)$, *then* (2.17) *holds for each sensed pair* p, q *of positive integers.*

PROOF. Suppose $(n_1, n_2, \ldots)$ is a sequence of positive integers such that $C_{pq} = 0$ if $p \geqslant n_q$ $(q = 1, 2, \ldots)$ and E is a matrix of positive numbers such that $|D(z)_{pq}| \leqslant E_{pq}$ for each z in Z $(p, q = 1, 2, \ldots)$. If q is a positive integer and $m = n_q$ then from (2.14) we have

$$|zD(z)_{pq} - I_{pq}| = |\sum_{r=1}^{m} D(z)_{pr}\, C_{rq}| \leqslant \sum_{r=1}^{m} E_{pr}\,|C_{rq}|, \qquad (2.22)$$

for each positive integer p and each z in Z. Hence we see that $D(z)_{pq} \to 0$ as $z \to \infty$ in Z and (2.15) and (2.17) hold (for all p and q), and the theorem is established.

Finally we have a theorem which will be of particular interest in the following section of this paper (see Corollary 3.2(a) and Theorem 3.4):

THEOREM 2.6. *If the matrix* A *is completely squarable and* $\operatorname{Im} w \leqslant 0$ *for all* w *in* $W(A)$, *and* $E(z)$ *is a matrix of functions—regular for* $\operatorname{Im} z > 0$*—such that, for* $\operatorname{Im} z > 0$, $E(z)$ *is completely squarable and* $\operatorname{Im} w \leqslant 0$ *for all* w *in* $W[E(z)]$, *then there is a matrix* $F(z)$ *of functions—regular for* $\operatorname{Im} z > 0$*—with the properties, for each* z *such that* $\operatorname{Im} z > 0$, *that* $F(z)$ *is a reciprocal of the matrix* $[zI - A - E(z)]$ *and*

$$(\operatorname{Im} z)\,\| F(z)_n x \|_n^2 + \operatorname{Im}(F(z)_n x, x)_n \leqslant 0, \qquad (2.23)$$

for each positive integer n *and each sequence* x *in* S.

PROOF. This theorem may be proved by proceeding as in the proof of Theorem 2.1 with the substitution of i for u, 0 for $v(u)$, and $A + E(z)$ for A, and with the observation that—for $\operatorname{Im} z > 0$—the reciprocal of the n-th segment of $[zI - A - E(z)]$ is an n-by-n matrix of regular functions. We omit the details.

It should be observed that the condition (2.23) on the matrix $F(z)$ is equivalent to the condition that

$$\| F(z)_n x - \frac{1}{z - z^*} x \|_n \leqslant \| \frac{1}{z - z^*} x \|_n \text{ for } \operatorname{Im} z > 0, \quad (2.24)$$

for each positive integer n and each x in S. Moreover, we observe that, for a symmetric matrix M, if s and t are *real* sequences in S and $x = s + it$, then

$$(M_n x, x)_n = (M_n s, s)_n + (M_n t, t)_n \text{ for } n = 1, 2, \ldots. \quad (2.25)$$

3. Non-decreasing matrices. A proof of the following theorem is contained in arguments given by Shohat and Tamarkin ([6], especially proof of Lemma 2.2, p. 25)—they do not state the theorem, but it will be assumed here without further argument :

THEOREM 3.1. *In order that the function $f(z)$ be regular and have non-positive imaginary part for* Im $z > 0$ *and* $z\,f(z)$ *be bounded as* Im $z \to +\infty$ *with* Re $z = 0$, *it is necessary and sufficient that there exist a bounded non-decreasing function $g(t)$ over the real numbers such that*

$$f(z) = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, dg(t) \quad \textit{for } \operatorname{Im} z > 0. \quad (3.1)$$

If $g(t)$ is a real function of bounded variation on the real numbers and is " normalized" so that

$$g(t) = [g(t+) + g(t-)]/2 \quad \text{for all } t, \quad (3.2)$$

and $f(z)$ is given by (3.1), then we have the Stieltjes inversion formula (see, for example, Wall [7], p. 250)

$$g(t) - g(t') = \frac{1}{\pi} \lim_{y \to 0+} \int_{t}^{t'} \operatorname{Im}\,[f(x+iy)]\, dx\,; \quad (3.3)$$

moreover, if $g_0 = g(+\infty) - g(-\infty)$, then for each positive number b less than 1 we have the asymptotic formulas

$$f(z) \to 0 \text{ as } z \to \infty \text{ with } \operatorname{Im} z \geqslant b, \text{ and} \quad (3.4)$$

$$zf(z) \to g_0 \text{ as } z \to \infty \text{ with } \operatorname{Im} z \geqslant b|z|. \quad (3.5)$$

A function $f(z)$ given by an integral of the form (3.1) is called the *Stieltjes transform* of the function $g(t)$. We have the following corollary to Theorem 3.1 :

COROLLARY 3.1(a). *If $f(z)$ is regular for* Im $z > 0$ *then, in order that $f(z)$ be the Stieltjes transform of a bounded non-*

decreasing function, it is necessary and sufficient that there exist a non-negative real number c such that

$$\left|f(z) - \frac{c}{z - z^*}\right| \leqslant \left|\frac{c}{z - z^*}\right| \text{ for } \operatorname{Im} z > 0; \tag{3.6}$$

if $f(z)$ is the Stieltjes transform of the bounded non-decreasing function $g(t)$, then $[g(+\infty) - g(-\infty)]$ is the least non-negative number c such that (3.6) *holds.*

Proof. Suppose $f(z)$ is regular for $\operatorname{Im} z > 0$. If c is a non-negative real number such that (3.6) holds, then we have

$$(\operatorname{Im} z)|f(z)|^2 + c \operatorname{Im} f(z) \leqslant 0 \text{ and } (\operatorname{Im} z)|f(z)| \leqslant c, \tag{3.7}$$

for $\operatorname{Im} z > 0$, so that—by Theorem 3.1 —$f(z)$ is the Stieltjes transform of a bounded non-decreasing function. Suppose, now, that $f(z)$ is the Stieltjes transform of the bounded non-decreasing function $g(t)$, and let $g(+\infty) - g(-\infty) = g_0$. Then g_0 is non-negative, and for $\operatorname{Im} z > 0$

$$f(z) - \frac{g_0}{z - z^*} = \frac{1}{z - z^*} \int_{-\infty}^{+\infty} \frac{t - z^*}{z - t} \, dg(t), \tag{3.8}$$

and this integrand has modulus 1 so that (3.6) holds with $c = g_0$. The last assertion of the Corollary follows from (3.5), and this completes the proof.

We now proceed to investigate a matric analogue of this theory of the Stieltjes transform—one of our aims being to shed more light on the theorems of the preceding section.

Definition 3.1. *A non-decreasing matrix is a matrix $G(t)$ of complex valued functions over the real numbers such that, for each positive integer n and each sequence x in S, the function $(G(t)_n x, x)_n$ is bounded and non-decreasing over the real numbers and for all t*

$$(G(t)_n x, x)_n = (G(t+)_n x, x)_n - (G(-\infty)_n x, x)_n; \tag{3.9}$$

a unit non-decreasing matrix is a non-decreasing matrix $G(t)$ such that $G(+\infty) = I$.

We observe that if $G(t)$ is a non-decreasing matrix then, for all real t, $G(t)$ is Hermitian, and—for each positive integer n and each x and y in S—the function $(G(t)_n x, y)_n$ is of bounded variation with total variation not greater than the non-negative square root of the number

$$(G(+\infty)_n x, x)_n \, (G(+\infty)_n y, y)_n.$$

For a non-decreasing matrix $G(t)$ we shall use G^0 to denote $G(+\infty)$. If $G(t)$ is a non-decreasing matrix and $h(t)$ is a complex valued function over the real numbers then by the integral $\int_{-\infty}^{+\infty} h(t)\, dG(t)$ we mean a matrix M such that

$$M_{pq} = \int_{-\infty}^{+\infty} h(t)\, dG(t)_{pq} \text{ for } p, q = 1, 2, \ldots. \tag{3.10}$$

It is noted that a necessary and sufficient condition that the non-decreasing matrix $G(t)$ should be the matric representation of a *Zerlegung der Einheit*, as introduced by von Neumann [4] (see, especially, Theorems 14, 15, and 18, and Definition 16—p. 91), is that $G^0 = I$ and $G(t)\, G(t) = G(t)$ for all t.

THEOREM 3.2. *If $F(z)$ is a matrix of complex-valued functions defined for* Im $z > 0$ *then, in order that, for each positive integer n and each sequence x in S, the function $(F(z)_n x, x)_n$ be regular and have non-positive imaginary part for* Im $z > 0$ *and $z(F(z)_n x, x)_n$ be bounded as* Im $z \to +\infty$ *with* Re $z = 0$, *it is necessary and sufficient that there exist a non-decreasing matrix $G(t)$ such that*

$$F(z) = \int_{-\infty}^{+\infty} \frac{1}{z - t}\, dG(t) \text{ for } \operatorname{Im} z > 0. \tag{3.11}$$

PROOF. Suppose $F(z)$ is a matrix of complex-valued functions defined for Im $z > 0$. If there is a non-decreasing matrix $G(t)$ such that $F(z)$ is given by (3.11) then, by Theorem 3.1, we see that $F(z)$ satisfies the stated conditions of regularity and boundedness. Suppose, on the other hand, that $F(z)$ satisfies these conditions. Let p and q be positive integers. From Theorem 3.1, we see that for each sensed pair x, y of complex numbers

there is a bounded non-decreasing function $g(t, x, y)$ such that, for all real t, $g(t, x, y) = g(t+, x, y) - g(-\infty, x, y)$ and, for $\operatorname{Im} z > 0$,

$$\int_{-\infty}^{+\infty} \frac{1}{z-t}\, dg(t, x, y) = x^* x F(z)_{pp} + x^* y F(z)_{pq} + y^* x F(z)_{qp} + \\ + y^* y F(z)_{qq}. \qquad (3.12)$$

Hence we see that for $\operatorname{Im} z > 0$

$$F(z)_{pq} = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, dh(t),\ F(z)_{qp} = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, dh(t)^*, \qquad (3.13)$$

where, for all real t,

$$2h(t) = [g(t, 1, 1) - g(t, 1, 0) - g(t, 0, 1)] - i[g(t, 1, i) - \\ - g(t, 1, 0) - g(t, 0, 1)]. \qquad (3.14)$$

Thus we see that there exists a matrix $G(t)$ of functions of bounded variation on the real numbers such that (3.11) holds, and such that, for each real t, $G(t) = G(t)^*$ and (3.9) holds for $n = 1, 2, \ldots$ and all x in S; hence the functions $(G(t)_n x, x)_n$ are real and bounded and—by the inversion formula (3.3)—are also non-decreasing. Thus $G(t)$ is a non-decreasing matrix, and the theorem is proved.

By analogy with terminology in the scalar case, we shall call $F(z)$, given by (3.11), the Stieltjes transform of $G(t)$. For $F(z)$ the Stieltjes transform of the non-decreasing matrix $G(t)$, we make the following remarks:

REMARK 1. By the decomposition (1.2) and the inversion formula (3.3), $F(z)$ determines $G(t)$ uniquely.

REMARK 2. For each positive number b less than 1 and each sensed pair p, q of positive integers

$$F(z)_{pq} \to 0 \text{ as } z \to \infty \text{ with } \operatorname{Im} z \geqslant b, \text{ and}$$

$$zF(z)_{pq} \to G^0_{pq} \text{ as } z \to \infty \text{ with } \operatorname{Im} z \geqslant b\,|z|. \qquad (3.15)$$

REMARK 3. If n is a positive integer and x is a sequence in S such that $(G_n^0 x, x)_n > 0$, then for $\operatorname{Im} z > 0$ and each positive number M, $\operatorname{Im}(F(z)_n x, x)_n$ is less than or equal to

$$-\frac{\operatorname{Im} z}{(\operatorname{Im} z)^2 + M^2}(G(\operatorname{Re} z + M)_n x - G(\operatorname{Re} z - M)_n x, x)_n \quad (3.16)$$

and so is negative.

REMARK 4. As in (3.8), we have the matric equation

$$F(z) - \frac{1}{z - z^*} G^0 = \frac{1}{z - z^*} \int_{-\infty}^{+\infty} \frac{t - z^*}{z - t} dG(t) \quad (3.17)$$

for $\operatorname{Im} z > 0$, and this yields the inequality

$$\left|(F(z)_n x - \frac{1}{z - z^*} G_n^0 x, y)_n\right| \leqslant \frac{[(G_n^0 x, x)_n (G_n^0 y, y)_n]^{1/2}}{|z - z^*|} \quad (3.18)$$

for all x and y in S and $n = 1, 2, \ldots\ldots$; if, in particular, $G^0 = I$ then (3.18) is equivalent to the condition (2.23).

COROLLARY 3.2(a). *If the matrices A and $E(z)$ are as in Theorem 2.6 and, in addition, each of the sums $\Sigma|E(z)_{rq}|^2$ $(q = 1, 2, \ldots)$ is bounded as* $\operatorname{Im} z \to +\infty$ *with* $\operatorname{Re} z = 0$, *and $F(z)$ is a matrix of functions—regular for* $\operatorname{Im} z > 0$*—such that, for each z such that* $\operatorname{Im} z > 0$, *$F(z)$ is a reciprocal of $[zI - A - E(z)]$, then the condition* (2.23) *is necessary and sufficient for $F(z)$ to be the Stieltjes transform of a unit non-decreasing matrix.*

PROOF. Suppose A and $E(z)$ satisfy the stated conditions and the matrix $F(z)$ of functions — regular for $\operatorname{Im} z > 0$ — is a reciprocal of $[zI - A - E(z)]$ in this domain. If $F(z)$ is the Stieltjes transform of a unit non-decreasing matrix then, by Remark 4 preceding this Corollary, we have condition (2.23) satisfied. Suppose now that $F(z)$ satisfies (2.23). By Theorem 3.2, there is a non-decreasing matrix $G(t)$ of which $F(z)$ is the Stieltjes transfoım. By Schwarz's inequality, for $\operatorname{Im} z > 0$

$$|zF(z)_{pq} - I_{pq}|^2 \leqslant [\Sigma|A_{rq}|^2 + \Sigma|E(z)_{rq}|^2]/(\operatorname{Im} z)^2 \quad (3.19)$$

for $p, q = 1, 2, \ldots$, so that, by Remark 2 following the proof of Theorem 3.2, $G^0 = I$ and the corollary is proved.

We note that if $E(z)$ is the Stieltjes transform of the non-decreasing matrix $H(t)$ and H^0 is bounded then $E(z)$ satisfies the conditions imposed in the Corollary 3.2(a)—moreover, the assumed half-boundedness of A and $E(z)$ is unnecessary in this proof.

COROLLARY 3.2(b). *If $F(z)$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ and A is a matrix such that*

$$z[zF(z)_{pq} - G^0_{pq}] \to A_{pq} \text{ as } \operatorname{Im} z \to \infty \text{ with } \operatorname{Re} z = 0, \quad (3.20)$$

for $p, q = 1, 2, \ldots$, then $\operatorname{Im} w \leqslant 0$ for all w in $W(A)$.

PROOF. Suppose $F(z)$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ and A is a matrix such that (3.20) holds for $p, q = 1, 2, \ldots$. Then for $\operatorname{Im} z > 0$ and each positive integer n and each x in S, the number

$$\operatorname{Im}(z^2F(z)_n x - zG^0_n x, x)_n = -(\operatorname{Im} z)\int_{-\infty}^{+\infty}\left|\frac{1}{z-t}\right|^2 d(G(t)_n x, x)_n \quad (3.21)$$

is non-positive—this completes the proof.

In anticipation of our next theorem, we state the following lemma which is due to Hamburger [1] (see the proof of Theorem IX, p. 268 ff) :

LEMMA A. *If $g(t)$ is a bounded non-decreasing function over the real numbers and $g_0 = g(+\infty) - g(-\infty)$ then, in order that $\int_{-\infty}^{+\infty} t\,dg(t)$ and $\int_{-\infty}^{+\infty} t^2dg(t)$ should exist, it is necessary and sufficient that there exist a real number h such that*

$$z^3\int_{-\infty}^{+\infty}\frac{1}{z-t}\,dg(t) - z^2g_0 - zh \quad (3.22)$$

is bounded as $\operatorname{Im} z \to +\infty$ with $\operatorname{Re} z = 0$; if (3.22) is so bounded then $h = \int_{-\infty}^{+\infty} t\,dg(t)$ and (3.22) $\to \int_{-\infty}^{+\infty} t^2dg(t)$.

THEOREM 3.3. *If $G(t)$ is a non-decreasing matrix, then in order that the sensed pair*

$$\int_{-\infty}^{+\infty} t\,dG(t),\ \int_{-\infty}^{+\infty} t^2dG(t) \quad (3.23)$$

of integrals should exist, it is necessary and sufficient that there exist an Hermitian matrix H such that for each sensed pair p,q of positive integers

$$z^3 \int_{-\infty}^{+\infty} \frac{1}{z-t}\, dG(t)_{pq} - z^2 G^0_{pq} - zH_{pq} \tag{3.24}$$

is bounded as $\operatorname{Im} z \to +\infty$ *with* $\operatorname{Re} z = 0$; *if H is an Hermitian matrix such that the expressions* (3.24) *are so bounded then*

(*i*) $H = \int_{-\infty}^{+\infty} t\, dG(t)$, *these integrals converging absolutely.*

(*ii*) *for each z such that* $\operatorname{Im} z > 0$, *we have*

$$z^2 \int_{-\infty}^{+\infty} \frac{1}{z-t}\, dG(t) = zG^0 + H + \int_{-\infty}^{+\infty} \frac{t^2}{z-t}\, dG(t).$$

(*iii*) *if* $K = \int_{-\infty}^{+\infty} t^2 dG(t)$ *then for each positive integer n and each x and y in S*

$$|(H_n x, y)_n|^2 \leqslant (K_n x, x)\,(G^0_n y, y)_n.$$

Proof. Suppose $G(t)$ is a non-decreasing matrix. If H is an Hermitian matrix such that the expressions (3.24) are bounded as $\operatorname{Im} z \to \infty$ with $\operatorname{Re} z = 0$, then we see by Lemma A that for each positive integer n and each x in S the sensed pair

$$\int_{-\infty}^{+\infty} t\, d(G(t)_n x, x)_n,\ \int_{-\infty}^{+\infty} t^2 d(G(t)_n x, x)_n \tag{3.25}$$

of integrals exist, $(H_n x, x)_n$ being the value of the first ; by the familiar decomposition

$$4\operatorname{Re}\,(H_n x, y)_n = (H_n[x+y], x+y)_n - (H_n[x-y], x-y)_n \tag{3.26}$$

and use of Schwarz's inequality, we then obtain the existence of (3.23) and the verification of statements (*iii*) and (*i*)—from which (*ii*) follows immediately. Similarly, if the integrals (3.23) exist, we find them absolutely convergent and, by Lemma A and the analogue of (3.26), we have the indicated boundedness of the expressions (3.24). This establishes the theorem.

Concerning a non-decreasing matrix $G(t)$ for which the integrals (3.23) exist, we make the following remarks;

Remark 1. The matrix $M(t)$ of complex-valued functions defined by the equation

$$M(t) = \int_{-\infty}^{t} s^2 \, dG(s) \tag{3.27}$$

is a non-decreasing matrix.

Remark 2. If G^0 is bounded then statements (i) and (iii) of Theorem 3.3 imply that H is squarable ; if $G^0 = I$ then the statement (iii) is equivalent to

$$\| H_n x \|^2 \leqslant (K_n x, x)_n, \text{ where } K = \int_{-\infty}^{+\infty} t^2 \, dG(t), \tag{3.28}$$

for x in S and $n = 1, 2, \ldots$.

Corollary 3.3(a). *If, for each positive integer n, G^n is an Hermitian matrix and $G(t)$ is a non-decreasing matrix with Stieltjes transform $F(z)$ then, in order that*

$$G^n = \int_{-\infty}^{+\infty} t^n \, dG(t) \text{ for } n = 1, 2, \ldots, \tag{3.29}$$

it is necessary and sufficient that, for each positive integer k and each sensed pair p,q of positive integers

$$z^{2k+1} F(z)_{pq} - z^{2k} G^0_{pq} - \ldots - z G^{2k-1}_{pq} \tag{3.30}$$

be bounded as $\operatorname{Im} z \to +\infty$ *with* $\operatorname{Re} z = 0$; *if* (3.29) *holds,*

(i) *the integrals* (3.29) *converge absolutely*;

(ii) *for* $\operatorname{Im} z > 0$ *and* $k = 1, 2, \ldots$, *we have the equation*

$$z^k F(z) = \sum_{n=0}^{k-1} z^{k-1-n} G^n + \int_{-\infty}^{+\infty} \frac{t^k}{z-t} \, dG(t);$$

(iii) *if* $G^0 = I$ *then, for* $k = 1, 2, \ldots$, G^k *is squarable and* $\| G^k_n x \|^2 \leqslant (G^{2k}_n x, x)_n$ *for x in S and* $n = 1, 2, \ldots$.

Proof. The proof is carried out by repeated application of Theorem 3.3 with a simple induction argument, observing Remarks 1 and 2 following that theorem. We omit the details.

Corollary 3.3(b). *If $F(z)$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ and, for* $\operatorname{Im} z > 0$ *and* $\operatorname{Re} z = 0$,

$F(z)$ is a reciprocal of $zI - H$ where H is a squarable Hermitian matrix, then $G^0 = I$ and

$$H = \int_{-\infty}^{+\infty} t\, dG(t),\ HH = \int_{-\infty}^{+\infty} t^2 dG(t). \tag{3.31}$$

PROOF. This is established by applying Theorem 3.3 and Theorem 2.4, with Z the positive imaginary axis and $b = 1$, and by comparing the bound obtained for (2.20) with the inequality (3.28).

For the unit non-decreasing matrix $G(t)$ where the integrals (3.23) are bounded matrices, we now give the analogue of Nevanlinna's continued fraction algorithm (see, for example, Wall [7], p. 318, Theorem 84.4) :

THEOREM 3.4. *If $F(z)$ is the Stieltjes transform of the unit non-decreasing matrix $G(t)$,*

$$A = \int_{-\infty}^{+\infty} t\, dG(t),\ B = \int_{-\infty}^{+\infty} t^2 dG(t),$$

and the matrix B is bounded, then there is a non-decreasing matrix $E(t)$ such that $E^0 = B - AA^*$ and, for Im $z > 0$,

$$zI - A - \int_{-\infty}^{+\infty} \frac{1}{z - t}\, dE(t)$$

is the bounded reciprocal of $F(z)$.

PROOF. Suppose $F(z)$ is the Stieltjes transform of the unit non-decreasing matrix $G(t)$,

$$A = \int_{-\infty}^{+\infty} t\, dG(t),\ B = \int_{-\infty}^{+\infty} t^2 dG(t),$$

and the matrix B is bounded. Let N be a non-negative number such that N^2 is the norm of B, i.e. N^2 is the least number M such that

$$\| Bx \| \leqslant M \| x \| \quad \text{for all } x \text{ in } S_0. \tag{3.32}$$

From Remark 2 following Theorem 3.3, we see that A is a bounded matrix with norm not greater than N. Each of A and B is Hermitian.

From Remark 3 following Theorem 3.2 we see that, for $\operatorname{Im} z > 0$ and $n = 1, 2, \ldots$, the determinant of the n-th segment of $F(z)$ is not zero. For each positive integer n and each z with positive imaginary part, let $C^n(z)$ be a matrix such that (i) the n-th segment of $C^n(z)$ is the reciprocal of the n-th segment of $F(z)$ and (ii) $C^n(z)_{pq} = 0$ if either $p > n$ or $q > n$, and let $D^n(z)$ be a matrix such that (i) the n-th segment of $D^n(z)$ is the n-th segment of the matrix $[zI - A - C^n(z)]$ and (ii) $D^n(z)_{pq} = 0$ if either $p > n$ or $q > n$.

Now, for each positive integer n, each of $C^n(z)$ and $D^n(z)$ is a matrix of functions regular for $\operatorname{Im} z > 0$; moreover, by Remark 4 following 3.2, $F(z)$ satisfies (2.23) so that for each x in S

$$\operatorname{Im}(D^n(z)_n x, x)_n = (\operatorname{Im} z)\,\|x\|_n^2 - \operatorname{Im}(C^n(z)_n x, x)_n \leqslant 0. \quad (3.33)$$

For $\operatorname{Im} z > 0$ and each positive integer n and each sensed pair p, q of positive integers less than $n + 1$, we have :

$$\sum_{r=1}^{n} D^n(z)_{pr} F(z)_{rq} = zF(z)_{pq} - \sum_{r=1}^{n} A_{pr} F(z)_{rq} - I_{pq},$$

and

$$\sum_{r=1}^{n} [zD^n(z)_{pr}]\,[zF(z)_{rq}]$$
$$= z[z^2 F(z)_{pq} - zI_{pq} - A_{pq}] - \sum_{r=1}^{n} A_{pr}\,[z^2 F(z)_{rq} - zI_{rq}]. \quad (3.34)$$

Suppose now that n is a positive integer and b is a positive number less than 1. By Theorem 3.3, there is a positive number d such that

$$\left|\sum_{r=1}^{n} [zD^n(z)_{pr}]\,[zF(z)_{rq}] - B_{pq} + \sum_{r=1}^{n} A_{pr} A_{rq}\right| \leqslant b, \quad (3.35)$$

and

$$|zF(z)_{pq} - I_{pq}| \leqslant b/n, \quad (3.36)$$

for $\operatorname{Im} z \geqslant d$, $\operatorname{Re} z = 0$, and $p, q = 1, \ldots, n$. Hence there exists a sensed pair $H(z), K(z)$ of n-by-n matrices of functions such that, for all such values of p, q, and z,

$$zD^n(z)_{pq} - \sum_{r=1}^{n} [zD^n(z)_{pr}]\,[K(z)_{rq} b/n]$$
$$= B_{pq} - \sum_{r=1}^{n} A_{pr} A_{rq} + bH(z)_{pq}, \quad (3.37)$$

and

$$|H(z)_{pq}| \leqslant 1 \text{ and } |K(z)_{pq}| \leqslant 1. \tag{3.38}$$

We now use a well-known property of n-by-n matrices: namely that, if

$$\sum_{p=1}^{n} \sum_{q=1}^{n} |k_{pq}|^2 \leqslant b^2, 0 < b < 1, \text{ and}$$

$$x_p - \sum_{q=1}^{n} k_{pq}x_q = y_p \text{ for } p = 1,\ldots, n$$

then $(y, y)_n \geqslant (1 - b)^2(x, x)_n$. Hence we have the sums $\sum_{r=1}^{n} |zD^n(z)_{pr}|^2$ $(p = 1,\ldots, n)$ bounded for $\operatorname{Im} z \geqslant d$ and $\operatorname{Re} z = 0$. Hence, by this boundedness and (3.34) and the identity

$$zD^n(z)_{pq} - \sum_{r=1}^{n} [zD^n(z)_{pr}]\,[zF(z)_{rq}]$$
$$= \sum_{r=1}^{n} [zD^n(z)_{pr}]\,[I_{rq} - zF(z)_{rq}], \tag{3.39}$$

we find that, for $p, q = 1,\ldots, n$ and $\operatorname{Re} z = 0$,

$$zD^n(z)_{pq} \to B_{pq} - \sum_{r=1}^{n} A_{pr}A_{rq} \text{ as } \operatorname{Im} z \to + \infty. \tag{3.40}$$

Thus, by Theorem 3.2 and Remark 2 following that theorem, for each positive integer n there is a non-decreasing matrix $E^n(t)$ of which $D^n(z)$ is the Stieltjes transform, and which has the property that

$$E^n(+ \infty)_{pq} = B_{pq} - \sum_{r=1}^{n} A_{pr}A_{rq} \text{ for } p, q = 1,\ldots, n \tag{3.41}$$

and $E^n(+ \infty)_{pq} = 0$ if either $p > n$ or $q > n$. Now, for each positive integer n and each x and y in S_0,

$$(\operatorname{Im} z)\,|(D^n(z)x, y)| \leqslant (E^n(+ \infty)x, x)^{1/2}(E^n(+\infty)y, y)^{1/2} \tag{3.42}$$

for $\operatorname{Im} z > 0$, and

$$(E^n(+ \infty)x, y) \to (Bx, y) - (Ax, Ay) \text{ as } n \to \infty. \tag{3.43}$$

Using Theorem 3.2, and proceeding as in the proof of Theorem 2.1, we find that there is a non-decreasing matrix $E(t)$ with Stieltjes transform $D(z)$ such that

$$C(z) = zI - A - D(z) \tag{3.44}$$

is a reciprocal of $F(z)$ for $\operatorname{Im} z > 0$, and for x in S_0

$$(E^0 x, x) \leqslant (Bx, x) - \| Ax \|^2. \tag{3.45}$$

Now it remains merely to show that $E^0 = B - AA$. Just as we obtained (3.34), we find that for $\operatorname{Im} z > 0$

$$zD(z) = [zD(z)][I - zF(z)] + z[z^2F(z) - zI - A] - - A[z^2F(z) - zI]; \tag{3.46}$$

moreover we have

$$zF(z) - I = \frac{1}{z}[A + \int_{-\infty}^{+\infty} \frac{t^2}{z-t} dG(t)] \tag{3.47}$$

and this matrix is bounded, with norm not greater than the number $[1 + N/(\operatorname{Im} z)] N/|z|$ for each z with positive imaginary part. Hence, by Theorem 3.3 and the continuity of bounded linear forms to which we have referred previously, for $\operatorname{Re} z = 0$ and $p, q = 1, 2, \ldots$,

$$zD(z)_{pq} \to B_{pq} - \Sigma A_{pr} A_{rq} \text{ as } \operatorname{Im} z \to + \infty. \tag{3.48}$$

This completes the proof.

It is to be noted that if $F(z)$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ and, for $\operatorname{Im} z > 0$, $F(z)$ is symmetric then, for each real t, $G(t)$ is a real symmetric matrix.

4. Application to J-fractions. In this section we consider the matrix J defined as follows : for each positive integer p

$$J_{pp} = a_p,\ J_{p,p+1} = J_{p+1,p} = b_p \neq 0, \text{ and}$$
$$J_{p,p+n} = J_{p+n,p} = 0 \text{ for } n = 2, 3, \ldots. \tag{4.1}$$

The continued fraction (J-fraction)

$$\frac{1}{z - a_1 -}\ \frac{b_1^2}{z - a_2 -}\ \frac{b_2^2}{z - a_3 - \ldots} \tag{4.2}$$

is associated with the matrix J in the following manner: we let $X_1(z), X_2(z), \ldots$ and $Y_1(z), Y_2(z), \ldots$ be sequences of polynomials such that $X_1(z) = 0$, $Y_1(z) = 1$, and for each positive integer n

$$\Sigma J_{nr} X_r(z) = I_{n1} + z X_n(z),\ \Sigma J_{nr} Y_r(z) = z Y_n(z); \tag{4.3}$$

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then, for each positive integer n, $X_{n+1}(z)/Y_{n+1}(z)$ is the n-th approximant of (4.2). The power series expansion

$$c_0/z + c_1/z^2 + c_2/z^3 + \dots \quad (c_0 = 1) \tag{4.4}$$

of the J-fraction (4.2) is defined by

$$X_{n+1}(z)/Y_{n+1}(z) = \sum_{r=0}^{2n-1} c_r/z^{r+1} + B_n(z)/z^{2n+1} \tag{4.5}$$

where $B_n(z)$ is bounded as $z \to \infty$ $(n = 1, 2, \dots)$. Let $J^{(1)} = J$ and $J^{(n+1)} = J^{(n)} J$ for $n = 1, 2, \dots$; then, for $n = 1, 2, \dots$, $J^n_{11} = c_n$. Let Y be a matrix such that, for each positive integer n, $\Sigma\, Y_{nr} z^{r-1} = Y_n(z)$; then we have the identities

$$\sum_{r=1}^{m} \sum_{s=1}^{n} Y_{mr}\, c_{r+s-2}\, Y_{ns} = I_{mn}, \text{ and}$$

$$\sum_{r=1}^{m} \sum_{s=1}^{n} Y_{mr}\, c_{r+s-1}\, Y_{ns} = J_{mn}. \tag{4.6}$$

If z is a complex number, then in order that the matrix R should be a reciprocal of $zI - J$ it is necessary and sufficient that, for $q \leqslant p$,

$$R_{pq} = R_{qp} = [Y_p(z)\, R_{11} - X_p(z)]\, Y_q(z), \quad (p, q = 1, 2, \dots); \tag{4.7}$$

we let $R[f, z]$ denote the matrix R defined by (4.7) with $R_{11} = f(z)$, and obtain the following theorem (for the equivalence of (i) and (ii) in this theorem, the reader is referred, for instance, to Wall [7], p. 316):

THEOREM 4.1. *If Z is a closed unbounded set in the plane and $f(z)$ is a complex valued function defined over Z, each two of the following statements are equivalent:*

(i) *for $n = 1, 2, \dots$*

$$z^n[f(z) - \sum_{r=0}^{n-1} c_r/z^{r+1}] \to 0 \text{ as } z \to \infty \text{ in } Z.$$

(ii) *for $n = 0, 1, 2, \dots$*

$$z^{2n}[f(z) - X_{n+1}(z)/Y_{n+1}(z)] \to 0 \text{ as } z \to \infty \text{ in } Z.$$

(iii) $R[f, z]_{pq} \to 0$ *as $z \to \infty$ in Z $(p, q = 1, 2, \dots)$.*

Proof. Suppose Z is a closed unbounded set in the plane and $f(z)$ is a complex valued function defined over Z. That (i) implies (ii) follows from (4.5) and the observation that for $n \geqslant 1$ the expression in (ii) equals

$$z^{2n}[f(z) - \sum_{r=0}^{2n-1} c_r/z^{r+1}] - B_n(z)/z.$$

That (ii) implies (iii) follows from the observation that, for $p \geqslant q$, $R[f, z]_{pq}$ is equal to

$$z^{2p-2}[f(z) - X_p(z)/Y_p(z)][Y_p(z)Y_q(z)/z^{2p-2}]$$

and $Y_p(z)$ is a polynomial of degree $p-1$. Suppose now that (iii) holds. We have, for $n = 1, 2, \ldots,$

$$R[f, z] = \frac{1}{z} I + \sum_{r=1}^{n-1} \frac{1}{z^{r+1}} J^{(r)} + \frac{1}{z^n} J^{(n)} R[f, z]$$

so that, in particular,

$$z^n[f(z) - \sum_{r=0}^{n-1} c_r/z^{r+1}] = \sum_{s=1}^{n+1} J^{(n)}_{1s} R[f, z]_{s1}\,.$$

Hence (i) holds, and the theorem is proved.

Wall and Wetzel [8] introduce the following definitions: (1) the J-fraction (4.2) is *positive definite* provided that if $\operatorname{Im} z > 0$ and w is in $W(zI - J)$ then $\operatorname{Im} w > 0$, i.e., $\operatorname{Im} w \leqslant 0$ for all w in $W(J)$; (2) the function $f(z)$, defined for $\operatorname{Im} z > 0$, is *asymptotically equal* to (4.2) provided that, for each positive number d, statement (ii) of Theorem 4.1 holds with Z the set $\operatorname{Im} z \geqslant d$. For (4.2) positive definite, they further define a class E of functions, regular for $\operatorname{Im} z > 0$, called *equivalent functions* of (4.2): they prove that if (4.2) is positive definite and $f(z)$ is in E then $f(z)$ is asymptotically equal to (4.2) and is the Stieltjes transform of a bounded non-decreasing function.

Let E' denote the class of functions $f(z)$, regular for $\operatorname{Im} z > 0$, such that for *real* x in S

$$(\operatorname{Im} z)\, \|R[f, z]_n x\|_n^2 + \operatorname{Im}(R[f, z]_n x, x)_n \leqslant 0,$$

$$\text{for } \operatorname{Im} z > 0 \text{ and } n = 1, 2, \ldots \qquad (4.8)$$

Let E'' denote the class of functions $f(z)$, regular for $\text{Im } z > 0$, such that (4.8) holds for *all* x in S. For (4.2) positive definite, Wall ([7], p. 232) shows that E is a subclass of E', and ([7], p. 318) conjectures that E is a proper subclass of E' and that for $f(z)$ to belong to E' it is necessary and sufficient that, for $\text{Im } z > 0$, $f(z)$ be regular and have negative imaginary part and be asymptotically equal to (4.2). We are now prepared to answer—in part—these conjectures.

THEOREM 4.2. *If $f(z)$ is a function defined for* $\text{Im } z > 0$, *then, in order that $f(z)$ should belong to E', it is necessary and sufficient that $R[f, z]$ be the Stieltjes transform of a non-decreasing matrix; if $R[f, z]$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ then $G^0 = I$, $f(z)$ belongs to E'', $f(z)$ is asymptotically equal to* (4.2), *and* (4.2) *is positive definite.*

PROOF. Suppose $f(z)$ is a function defined for $\text{Im } z > 0$. If $R[f, z]$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ then

(i) by Theorem 4.1 and Remark 2 following Theorem 3.2, $f(z)$ is asymptotically equal to (4.2),

(ii) since $zR[f, z] = I + JR[f, z]$ for $\text{Im } z > 0$ then, by Remark 2 following Theorem 3.2, $G^0 = I$,

(iii) by Remark 4 following Theorem 3.2, $f(z)$ is in E'', and

(iv) by Corollary 3.2(b), (4.2) is positive definite.

Suppose now that $f(z)$ belongs to E'. Then—for $\text{Im } z > 0$ and $n = 1, 2, \ldots$ —the inequalities

$$\text{Im}(R[f, z]_n x, x)_n \leqslant 0 \text{ and } \| (\text{Im } z) R[f, z]_n x \|_n \leqslant \| x \|_n, \quad (4.9)$$

are found to hold for real x in S, the latter by the intermediate step of obtaining (2.24) for $F(z) = R[f, z]$. Since $R[f, z]$ is symmetric then, by the observation at the end of section 2 of this paper, the first of the inequalities (4.9) holds for all x in S.

The second of these inequalities shows, in particular, that when $\operatorname{Im} z > 0$ and $\operatorname{Re} z = 0$,

$$\sum_{p=1}^{n} |zR[f, z]_{pq}|^2 \leqslant 1 \text{ for } n \geqslant q = 1, 2, \ldots. \tag{4.10}$$

Hence, by Theorem 3.2, $R[f, z]$ is the Stieltjes transform of a non-decreasing matrix, and the theorem is proved.

We note that the existence of a function in E' implies that (4.2) is positive definite, and that each of the following conditions is necessary and sufficient for the function $f(z)$ to belong to E':

CONDITION 1. $f(z)$ belongs to E''.

CONDITION 2. $f(z)$ is regular for $\operatorname{Im} z > 0$, is asymptotically equal to (4.2), and is such that $\operatorname{Im} w \leqslant 0$ for w in $W(R[f,z])$ with $\operatorname{Im} z > 0$.

CONDITION 3. $R[f,z]$ is the Stieltjes transform of a unit non-decreasing matrix.

Suppose now that (4.2) is positive definite. For each positive integer n, let k_n denote the minimum of $\operatorname{Im}(-J_n x, x)_n$ for all x in S such that $|x_n| = 1$. By (2.25) we see that, for each positive integer n, k_n is the minimum of $\operatorname{Im}(-J_n x, x)_n$ for all *real* x in S such that $x_n^2 = 1$.

With a characterization by Wall ([7], p. 86 *ff.*) of the minimal parameters of a chain sequence, we can define the class E as follows: $f(z)$ belongs to E provided that—for $\operatorname{Im} z > 0$ — $f(z)$ is regular, the function $f_n(z)$ defined as

$$b_n[Y_{n+1}(z)\, f(z) - X_{n+1}(z)]/[Y_n(z)\, f(z) - X_n(z)]$$

is regular ($n = 1, 2, \ldots$), and

$$\operatorname{Im}[f_n(z)] \leqslant k_n + \operatorname{Im} z \quad \text{for } n = 1, 2, \ldots.$$

This is a paraphrase of the definition given by Wall and Wetzel [8] and by Wall ([7], p. 231), who consider the functions

$$w_n(z) = b_n^2/f_n(z) = z - a_{n+1} - f_{n+1}(z)$$

and require that $\operatorname{Im}[w_n(z)] \geqslant \operatorname{Im}(-a_{n+1}) - k_{n+1}$.

Suppose, for the moment, that $f(z)$ belongs to E'. Then $R[f, z]$ is the Stieltjes transform of a unit non-decreasing matrix. Hence, for $\mathrm{Im}\, z > 0$ and $n = 1, 2, \ldots$, $f_n(z)$ is regular and the determinant of the n-th segment of $R[f, z]$ is not zero (see Remark 3 following Theorem 3.2); moreover we find that for all x in S (compare Wall [7], p. 232)

$$z \| R[f, z]_n x \|_n^2 - (R[f, z]_n x, x)_n^* = (J_n R[f, z]_n z, R[f, z]_n x)_n + f_n(z) \left| \sum_{q=1}^{n} R[f, z]_{nq} x_q \right|^2,$$

and this has non-positive imaginary part since E' is E''. But, since the determinant of the n-th segment of $R[f, z]$ is not zero, this last statement is equivalent to

$$\mathrm{Im}[f_n(z)] \leqslant k_n \quad (\mathrm{Im}\, z > 0 \text{ and } n = 1, 2, \ldots).$$

Hence $f(z)$ is in E. Since, as noted previously, Wall [7] has shown that E is a subclass of E', we now have:

THEOREM 4.3. *E' is E.*

It should be noted that the class E is the class of functions considered by Hellinger and Wall [3] in their early investigation concerning the generalization of real J-fractions to (4.2) with the assumption that $\mathrm{Im}\, a_p \leqslant 0$ and b_p is real. They ([3], Theorem 8.2.) obtain Stieltjes integral representations for the functions in E by proving that a sufficient condition (which is *not necessary*) for $f(z)$, regular in $\mathrm{Im}\, z > 0$, to be the Stieltjes transform of a non-decreasing functian $g(t)$ with $g(+\infty) - g(-\infty) = 1$, is that there exist a constant C such that for $\mathrm{Im}\, z > 0$,

$$f(x) - \frac{1}{z - z^*} = \frac{1}{z - z^*}\left[\frac{h(z) - z^*}{z}\right], \quad |h(z)| \leqslant C. \quad (4.11)$$

This is the theorem used by Wall and Wetzel [8], and it would seem that our Corollary 3.1(a) provides a natural extension of that theorem.

Concerning the case when the matrix J is real, we make certain observations. If $R[f, z]$ is the Stieltjes transform of the non-decreasing matrix $G(t)$ then, by Corollary 3.3(a),

$$J^{(n)} = \int_{-\infty}^{+\infty} t^n dG(t) \text{ for } n = 1, 2, \ldots.$$

Suppose, on the other hand, that $g(t)$ is a bounded non-decreasing function over the real numbers such that $g(t) = g(t+) - g(-\infty)$ for all t, $g(+\infty) = 1$, and

$$\int_{-\infty}^{+\infty} t^n dg(t) = c_n \text{ for } n = 1, 2, \dots .$$

Let $f(z)$ be the Stieltjes transform of $g(t)$, and let the matrix $G(t)$ be defined by

$$G(t)_{pq} = \int_{-\infty}^{t} Y_p(u)\, Y_q(u)\, dg(u) \text{ for } p,q = 1, 2, \dots . \tag{4.12}$$

Then, by (4.6), $G(t)$ is a unit non-decreasing matrix, with Stieltjes transform $F(z)$ such that for $p, q = 1, 2, \dots$.

$$F(z)_{pq} = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, Y_p(t)\, Y_q(t) dG(t).$$

With the observation that $Y_1(t) = 1$ and

$$zY_n(t) - \Sigma J_{nr} Y_r(t) = [z - t]\, Y_n(t) \text{ for } n = 1, 2, \dots .$$

we see that $F(z) = R[f, z]$.

We have not succeeded in finding an analogue of (4.12) in the case of the general positive definite J-fraction, although—by the note at the end of section 3 of this paper—$G(t)$ must be real and symmetric even in the general case.

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ON THE MAXIMUM MODULUS AND THE COEFFICIENTS OF AN ENTIRE SERIES

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1. Let $f(z) = \sum_{0}^{\infty} a_n z^n$ be an entire function, $\mu(r)$ the maximum term of the series for $|z| = r$ and $\nu(r)$ its rank. A number of results connecting $M(r)$, $\mu(r)$, $\nu(r)$ and a_n are known [4, pp. 28-46, 93-106 ; 2, pp. 3-11 ; 3 ; 5] but some of them are true only for functions of finite order. In this paper we prove several relations between these auxiliary functions $M(r)$ etc., which are true whether $f(z)$ be of finite or infinite order. We shall use the following notations.

$$\begin{matrix} T_k \\ t_k \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k M(r)}{l_k r}, \quad \begin{matrix} P_k \\ \rho_k \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k M(r)}{l_{k-1} r};$$

$$\begin{matrix} N_k \\ \nu_k \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k-1} \nu(r)}{l_{k-1} r}, \quad \begin{matrix} \Delta_k \\ \delta_k \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k-1} \nu(r)}{l_k r};$$

$$\begin{matrix} A_k \\ \alpha_k \end{matrix} = \lim_{n\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k-1} n}{l_{k-1}\left(\frac{1}{n}\log\frac{1}{|a_n|}\right)},$$

$$\begin{matrix} B_k \\ \beta_k \end{matrix} = \lim_{n\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k-1} n}{l_{k-2}\left(\frac{1}{n}\log\frac{1}{|a_n|}\right)},$$

$$\begin{matrix} S_k \\ \sigma_k \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k \mu(r)}{l_k r}, \quad \begin{matrix} F_k \\ \phi_k \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k \mu(r)}{l_{k-1} r},$$

where k is any (fixed) integer $\geqslant 2$ and $l_k r$ denotes the k-th iterate of $\log r$ [1, p. 16].

2. **Theorem** 1. *For any entire function we have*

$$t_k = \sigma_k \qquad (k = 2, 3, \ldots) \tag{2.1}$$

$$T_k = S_k \qquad ,, \tag{2.2}$$

$$\delta_k \geqslant \alpha_k \qquad (k = 2, 3, \ldots) \tag{2.3}$$
$$\Delta_k = A_k \qquad ,, \tag{2.4}$$
$$t_2 = 1 + \delta_2 \tag{2.5}$$
$$T_2 = 1 + \Delta_2 \tag{2.6}$$
$$t_k = \max(1, \delta_k) \qquad (k = 3, 4, \ldots) \tag{2.7}$$
$$T_k = \max(1, \Delta_k) \qquad ,, \tag{2.8}$$
$$\rho_k = \phi_k = \nu_k \geqslant \beta_k \qquad (k = 2, 3, \ldots) \tag{2.9}$$
$$P_k = F_k = B_k = N_k \qquad ,, \tag{2.10}$$

THEOREM 2. *If* $|a_n/a_{n+1}| \uparrow$ *for all large* n, *then for* $k = 2, 3, \ldots,$

$$\rho_k = \beta_k\,, \tag{2.11}$$
$$\alpha_k = \delta_k\,. \tag{2.12}$$

3. (i) Proof of $t_2 = \sigma_2$.

Since $M(r) \geqslant \mu(r)$ we have

$$t_k \geqslant \sigma_k \geqslant 1, \qquad (k \geqslant 2)\,. \tag{3.1}$$

To prove $t_2 \leqslant \sigma_2$, we may suppose $\sigma_2 < \infty$. If $a > \sigma_2$, we have

$$\log \mu(r) < (\log r)^a,$$

for a sequence of values of $r \to \infty$. Further for $r > r_0(f)$

$$\frac{\nu(\sqrt{r}) \log r}{2} < \log \mu(r)\,. \tag{3.2}$$

Hence

$$\liminf_{r\to\infty} \frac{\log \nu(r)}{l_2 r} \leqslant a - 1.$$

Choose numbers β, γ such that $a - 1 < \beta$, $(a-1)/\beta < \gamma < 1$. Then we have $\log \nu(r) < \beta\gamma l_2 r$ for a sequence $r = x_n$ (say), $x_n \uparrow \infty$. Let $X_n = \exp\{(\log x_n)^\gamma\}$ and I_n denote the interval $X_n \leqslant r \leqslant x_n$. Then for every r of I_n

$$\begin{aligned}\log \nu(r) \leqslant \log \nu(x_n) &< \beta\, \gamma\, l_2\, x_n \\ &= \beta \log (\log x_n)^\gamma \leqslant \beta\, l_2\, r.\end{aligned}$$

Take $Y_n = 1 + X_n$. Then for all large n, Y_n and $2Y_n$ lie inside I_n and we have [4, p. 32]

$$\log M(Y_n) < \log \mu(Y_n) + \log \nu(2Y_n) + O(1),$$

Further

$$\log \nu(2Y_n) = O(l_2 Y_n)$$

$$\log \mu(Y_n) \geqslant \log Y_n, \text{ for } n > n_1(f).$$

Hence

$$\log M(Y_n) < (1 + o(1)) \log \mu(Y_n)$$

$$l_2 M(Y_n) < O(1) + \log \nu(Y_n) + l_2 Y_n$$

$$< O(1) + (\beta + 1)\, l_2 Y_n.$$

Hence $t_2 \leqslant \sigma_2$ and so $t_2 = . \sigma_2$.

(ii) $t_2 = 1 + \delta_2$.

To prove $t_2 \leqslant 1 + \delta_2$, we suppose $\delta_2 < \infty$. Let

$$\delta_2 < B < \beta, \frac{B}{\beta} < \gamma < 1.$$

Then we get as in (i)

$$l_2 M(r) < O(1) + (\beta + 1) l_2\, r,$$

for a sequence of values of $r \to \infty$. Hence $t_2 \leqslant 1 + \delta_2$. From (3.2) we get $t_2 \geqslant 1 + \delta_2$ and (ii) follows.

(iii) $t_2 \geqslant 1 + \alpha_2$.

We suppose $\alpha_2 > 0$. Consider first $\alpha_2 < \infty$. If $0 < \alpha < \alpha_2$ we have

$$|a_n| > \exp(-n^{1+1/\alpha}), \ n \geqslant N = N(\alpha).$$

Let

$$r_n = \exp(n^{1/\alpha} + n^{1/\alpha_2}), \ r_n \leqslant r \leqslant r_{n+1}$$

Then from Cauchy's inequality

$$M(r) \geqslant |a_n| r^n \geqslant |a_n|\, r_n^n$$

$$> \exp(-n^{1+1/\alpha}) r_n^n.$$

Hence

$$\log M(r) \geqslant n^{1+1/\alpha_2}.$$

Further

$$l_2 r \leqslant l_2 (r_{n+1}) \sim \frac{1}{\alpha} \log (n + 1)$$

and hence

$$t_2 \geqslant 1 + \alpha_2.$$

If $\alpha_2 = \infty$, the above argument gives $t_2 = \infty$.

(iv) $\sigma_k = \max(1, \delta_k)$, $(k \geqslant 3)$.

Since $\nu(r/2) \log 2 < \log \mu(r)$ it follows that $\delta_k \leqslant \sigma_k$ and hence from (3.1), $\max(1, \delta_k) \leqslant \sigma_k$.

If $\delta_k < 1$, let $\delta_k < a < 1$; then

$$l_{k-1}\, \nu(r) < a\, l_k\, r,$$

for a sequence of values of $r \to \infty$. Further

$$\begin{aligned} l_2\, \mu(r) &< \log \nu(r) + l_2\, r + o(1) \\ &< 2\, l_2\, r + o(1), \end{aligned} \tag{3.3}$$

for a sequence of values of $r \to \infty$. Hence $\sigma_k \leqslant 1$ and so $\sigma_k = 1$. If $1 \leqslant \delta_k < \infty$, let $a > \delta_k$. From (3.3)

$$l_2\, \mu(r) < 3\, e_{k-2}\,(a\, l_k\, r)$$

for a sequence of values of $r \to \infty$. Hence $\sigma_k \leqslant \delta_k$ which holds if $\delta_k = \infty$. Hence (iv) follows.

(v) From (ii) and (iii) it follows that $\delta_2 \geqslant \alpha_2$. By an argument similar to that given in (i) we can show that $t_k = \sigma_k$, $(k \geqslant 3)$.

We omit the proofs of (2.2), (2.3), (2.4), (2.6) and (2.8). The relation (2.7) follows from (2.1) and (iv).

The relations (2.9), (2.10), (2.11), for $k = 2$, are known [4, pp. 32-34 ; 5 ; 3(i)]. The proofs of (2.9) and (2.10), for $k \geqslant 3$, are similar to those for $k = 2$ and are omitted.

4. PROOF OF (2.12). Let

$$F(z) = \sum_0^\infty C\,|a_n|\, z^n + P(z),$$

where $P(z)$ is a polynomial and C a non-zero constant (not involving n). Then

$$\delta_k = \delta_k(f) = \delta_k(F);\ \alpha_k = \alpha_k(f) = \alpha_k(F).$$

Hence we may suppose that

$$f(z) = 1 + \sum_1^\infty \frac{z^n}{R_1\, R_2 \ldots R_n}$$

where $0 < R_1 \leqslant R_2 \ldots \leqslant R_n \to \infty$ with n. Further

$$\delta_k = \liminf_{n\to\infty} \frac{l_{k-1}\, n}{l_k\, R_n}\,;\; a_k = \liminf_{n\to\infty} \frac{l_{k-1}\, n}{l_{k-1}\left(\dfrac{1}{n}\sum_{1}^{n}\log R_i\right)}.$$

Since $(R_1 R_2 \ldots R_n)^{1/n} \leqslant R_n$ we get $a_k \geqslant \delta_k$. To prove $a_k \leqslant \delta_k$ we suppose $\delta_k < \infty$. If $\delta_k < B$ then

$$\log R_n > e_{k-1}\left\{\frac{1}{B}(l_{k-1}\, n)\right\},$$

for an infinity of $n = n_p$ (say). Hence

$$\sum_{1}^{2n_p} \log R_\iota > \sum_{n_p}^{2n_p} \log R_\iota > n_p\, e_{k-1}\left\{\frac{1}{B} l_{k-1}\, n_p\right\}.$$

Hence for an infinity of n,

$$\frac{1}{2n}\log |a_{2n}| > \frac{1}{2}\left\{e_{k-1}\left(\frac{1}{B} l_{k-1}\, n\right)\right\},$$

$$l_{k-1}\left(\frac{1}{2n}\log\frac{1}{|a_{2n}|}\right) > \frac{1}{B} l_{k-1}(2n) + O(1),$$

and hence $a_k \leqslant \delta_k$ and (2.12) follows. We omit the proof of (2.11) for $k \geqslant 3$.

5. Finally we construct a function for which $|a_n/a_{n+1}|$ is an increasing function of n and for which $\Delta_k > \delta_k$.

Let $n_1, n_2, \ldots$ be a sequence of positive integers such that

$$n_1 = [e_k\, 2]!$$

$$n_{s+1} = [e_{k-1}\{(l_{k-1}\, n_s)^2\}],\; s = 1, 2, \ldots,$$

and let

$$R_n = 1 + \frac{n-1}{[R_{n_1} + n_2]!},\; n = 1, 2, \ldots, n_1 - 1;$$

$$= e_k\{(l_{k-1}\, n_s)^2\},\; n = n_s$$

$$= R_{n_s} + \frac{n - n_s}{[R_{n_s} + n_{s+1}]!}$$

$$(n = n_s + 1,\, n_s + 2, \ldots, n_{s+1} - 1;\; s = 1, 2, \ldots),$$

$$f(z) = \sum_{1}^{\infty} \frac{z^n}{R_1\, R_2 \ldots R_n}.$$

Then $f(z)$ is an entire function of zero order for which

$$\delta_k = 0, \ \Delta_k = 1.$$

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A QUASI-TAUBERIAN THEOREM ON FOURIER SERIES

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1. Introduction. The object of this note is to prove a theorem on J_μ-summability and to deduce from it a quasi-tauberian theorem on Fourier trigonometric series, which is a completion of a result due to C. T. Loo [7, Theorem 2]. The method of proof is very similar to the one adopted by K. Chandrasekharan [3].* In the course of the proof, a result due to L. Schmetterer [8] is also obtained as a special case.

2. Notations and preliminary lemmas. Let $J_\mu(t)$ denote the Bessel function of the first kind, of order μ, Re $\mu > -1/2$; write $\alpha_\mu(t) \equiv 2^\mu \Gamma(1+\mu)\, t^{-\mu} J_\mu(t)$. We have the following

DEFINITION. *A series* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* J_μ *to the sum* s, *if* $\sum_{\nu=0}^{\infty} a_\nu\, \alpha_\mu(\nu t)$ *converges for small positive* t, *and if*

$$\lim_{t\to+0} \sum_{\nu=0}^{\infty} a_\nu\, \alpha_\mu(\nu t) = s.$$

For properties of J_μ-summability see [3]. It is well known that as $t \to \infty$,

$$\alpha_\mu(t) = O(t^{-\mu-1/2})\,; \tag{2.1}$$

more generally, if we define $\alpha_\mu^r(t) \equiv \dfrac{d^r}{dt^r}\alpha_\mu(t)$, $r = 0, 1, 2, \ldots$, then as $t \to \infty$,

$$\alpha_\mu^r(t) = O(t^{-\mu-1/2}). \tag{2.1}$$

LEMMA 1. *If* $p > 0$, $q = [p]$ *and* $I(z)$ *denotes the integral*

$$I(z) = \int_1^\infty (y-1)^{q-p}\, \alpha_\mu^{q+2}(yz)\, dy, \tag{2.2}$$

*I wish to thank Professor K. Chandrasekharan for kindly suggesting this problem and for his help and criticism throughout the preparation of this note.

then

$$I(z) = \begin{cases} O(z^{p-\mu-q-3/2}), \; if\, |z| \geqslant 1; \\ O(z^{p-q-1}), \; if\, |z| < 1. \end{cases} \tag{2.3}$$

This is proved by K. Chandrasekharan [3, p. 225].

$$\text{Let } S_n = \sum_{\nu=0}^{n} a_\nu,\; S^0(x) \equiv S(x) = \sum_{\nu=0}^{n} a_\nu \text{ where } n \leqslant x < n+1,$$

$$S^p(x) = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} S(t)\, dt, \quad p > 0,$$

the last integral existing for all $p > 0$. We quote the following well-known theorem of M. Riesz [5, Theorem 1.71].

LEMMA 2. *Let $V(x)$ and $W(x)$ be two non-decreasing positive functions such that for any positive value of x, $S(x) = O(V(x))$ and $S^p(x) = O(W(x))$. Then for any r such that $0 < r < p$,*

$$S^r(x) = O\{(V(x))^{1-r/p}(W(x))^{r/p}\}.$$

3. Theorems on J_μ-summability. In this section we state two theorems on the J_μ-summability of series in general.

THEOREM I. *If*

$$a_n = O(n^{-\delta}),\; 0 < \delta < 1, \tag{3.1}$$

and

$$n^{-p} S^p(n) - s = O(n^{-\alpha}),\; p > \alpha, \tag{3.2}$$

then $\sum_{\nu=0}^{\infty} a_\nu$ is summable J_μ to the sum s for $\mu = p - 1/2$, provided that

$$0 < 1 - \alpha < (p + \delta - 1)/p. \tag{3.3}$$

PROOF. Without loss of generality, we may take $s = 0$. We first prove that $\sum_{\nu=0}^{\infty} a_\nu \alpha_{p-1/2}(\nu t)$ converges for small positive t, and then prove that $\sum_{\nu=0}^{\infty} a_\nu \alpha_{p-1/2}(\nu t) = o(1)$ as $t \to +0$. Now

$$\sum_{\nu=0}^{\infty} a_\nu \alpha_{p-1/2}(\nu t) = \Big(\sum_{\nu=0}^{n} + \sum_{\nu=n+1}^{\infty}\Big) a_\nu \alpha_{p-1/2}(\nu t) \equiv \phi_1 + \phi_2,$$

say, where n is to be chosen presently. By (2.1) and (3.1) we have

$$\phi_2 = \sum_{\nu=n+1}^{\infty} O(\nu^{-\delta}(\nu t)^{-p})$$
$$= O(n^{-p-\delta+1}\, t^{-p}), \tag{3.4}$$

provided that $p > 1 - \delta$, which is ensured by (3.3). This shows that for fixed $t > 0$, the series $\sum_{\nu=0}^{\infty} a_\nu\, \alpha_{p-1/2}(\nu t)$ converges for such p. Let

$$\phi(t) = \sum_{\nu=0}^{\infty} a_\nu\, \alpha_{p-1/2}(\nu t).$$

Now set $n = [t^{-r}]$, where r is chosen so as to satisfy

$$p/(p + \delta - 1) < r < 1/(1 - \alpha). \tag{3.5}$$

The existence of such an r is implied by (3.3). Then from (3.4) it follows that, as $t \to +0$,

$$\phi_2 = O(t^{r(p+\delta-1)-p}) = o(1). \tag{3.6}$$

Next, by Abel's lemma on partial summation, we obtain

$$\phi_1 = \sum_{\nu=0}^{n} a_\nu\, \alpha_{p-1/2}(\nu t) = \sum_{\nu=0}^{n-1} S_\nu[\alpha_{p-1/2}(\nu t) - \alpha_{p-1/2}(\overline{\nu+1}.\, t)] + $$
$$+ S_n \alpha_{p-1/2}(nt)$$
$$= -\sum_{\nu=0}^{n-1} \int_\nu^{\nu+1} S^0(x) \frac{d}{dx} \alpha_{p-1/2}(xt)\, dx + S_n\, \alpha_{p-1/2}(nt)$$
$$= -\int_0^n S^0(x) \frac{d}{dx} \alpha_{p-1/2}(xt)\, dx + S_n\, \alpha_{p-1/2}(nt).$$

Let $h = [p]$. Then integrating the first term in the last expression by parts $(h + 1)$ times, and writing $D_n^r \equiv \left[\frac{d^r}{dx^r}\right]_{x=n}$, we get

$$\phi_1 = S_n\, \alpha_{p-1/2}(nt) + \sum_{r=1}^{h} (-)^r\, S^r(n)\, D_n^r\, \alpha_{p-1/2}(xt) +$$
$$+ (-)^{h+1}\, S^{h+1}(n)\, D_n^{h+1}\, \alpha_{p-1/2}(xt) +$$
$$+ (-)^{h+2} \int_0^n S^{h+1}(x) \frac{d^{h+2}}{dx^{h+2}} \alpha_{p-1/2}(xt)\, dx$$
$$\equiv \phi_3 + \phi_4 + \phi_5 + \phi_6,$$

say. We now estimate the ϕ's separately. Since (3.1) trivially implies that $S_n = O(n^{1-\delta})$, it is easily seen that, as $t \to +0$,

$$\phi_3 = o(1), \tag{3.7}$$

because of (3.5). To estimate ϕ_4 we first observe that

$$S^r(n)\, D_n^{\nu}\, a_{p-1/2}(xt) = S^r(n)\, t^{\nu} \left[\frac{d^{\nu}}{d(tx)^{\nu}}\, a_{p-1/2}(xt) \right]_{x=n}.$$

Here $S^r(n)$ can be estimated by using Lemma 2, with $V(x) = x^{1-\delta}$ and $W(x) = x^{p-a}$, both increasing functions in view of (3.1) and (3.2), while $D_n^{\nu} a_{p-1/2}(xt)$ can be estimated by using (2.1). Thus we have

$$\begin{aligned}
\phi_4 &= \sum_{\nu=1}^{h} O(n^{(1-\delta)(1-\nu/p)+\nu(p-a)/p}\, t^{\nu}(nt)^{-p}) \\
&= \sum_{\nu=1}^{h} O(t^{rp-r(1-\delta)(1-\nu/p)-r\nu(p-a)/p+\nu-p}) \\
&= o(1),
\end{aligned} \tag{3.8}$$

as $t \to +0$, provided that

$$1 - \delta - p - \nu\,(1 - \delta - p + a)/p + (p-\nu)/r < 0,$$

i.e.

$$r > p(p-\nu)/\,\{(p-\nu)\,(p+\delta-1) + \nu a\} = r',$$

say, where $\nu = 1, 2, \ldots, h$; and since $r' < p/(p+\delta-1)$, as can be easily verified, it follows that (3.8) is secured by the choice of r as in (3.5).

We next take up ϕ_6 which gives the crucial estimate. Omitting constant factors, we have

$$\begin{aligned}
\phi_6 &= \int_0^n S^{h+1}(x)\, \frac{d^{h+2}}{dx^{h+2}}\, a_{p-1/2}(xt)\, dx \\
&= \int_0^n S^p(u)\, du \int_u^n (x-u)^{h-p}\, \frac{d^{h+2}}{dx^{h+2}}\, a_{p-1/2}(xt)\, dx \\
&= \int_0^n S^p(u)\, du \int_u^{\infty} (x-u)^{h-p}\, \frac{d^{h+2}}{dx^{h+2}}\, a_{p-1/2}(xt)\, dx - \\
&\qquad - \int_0^n S^p(u)\, du \int_n^{\infty} (x-u)^{h-p}\, \frac{d^{h+2}}{dx^{h+2}}\, a_{p-1/2}(xt)\, dx \\
&\equiv \psi_1 + \psi_2,
\end{aligned}$$

say. We first estimate ψ_2. For this we argue as in [3, p. 223], and using (3.2) obtain

$$\begin{aligned}\psi_2 &= O\,(n^{-p}\, n^{h-p}\, n^{p-a+1}\, t^{h+1-p}) \\ &= O\,(t^{r(p-h-1+a)+h+1-p}) \\ &= o\,(1), \qquad (3.9)\end{aligned}$$

as $t \to +0$, because of (3.5). On the other hand, by (2.2),

$$\begin{aligned}\psi_1 &= t^{h+2}\int_0^n S^p(u)\, u^{h+1-p}\, du \int_1^\infty (y-1)^{h-p}\, a_{p-1/2}^{h+2}\,(yut)\, dy \\ &= t^{h+2}\int_0^n S^p(u)\, u^{h+1-p}\, I\,(ut)\, du \\ &= t^{h+2}\left\{\int_0^{1/t} + \int_{1/t}^n\right\} = \chi_1 + \chi_2,\end{aligned}$$

say.

Now by Lemma 1 with $\mu = p - 1/2$, we get

$$\begin{aligned}\chi_1 &= t^{h+2}\int_0^{1/t} O\,\{u^{p-a}\, u^{h+1-p}\,(ut)^{p-h-1}\}\, du \\ &= t^{p+1}\int_0^{1/t} O\,(u^{p-a}) = O\,(t^a) \\ &= o(1), \qquad (3.10)\end{aligned}$$

as $t \to +0$. On the other hand

$$\begin{aligned}\chi_2 &= t^{h+2}\int_{1/t}^n O(u^{p-a}\, u^{h+1-p}\,(ut)^{-h-1})\, du \\ &= t\int_{1/t}^n O(u^{-a})\, du = O(t^a) + O(t^{1-r(1-a)}) \\ &= o(1), \qquad (3.11)\end{aligned}$$

as $t \to +0$, by (3.5). Combining (3.9), (3.10) and (3.11), we obtain, as $t \to +0$,

$$\phi_6 = o(1). \qquad (3.12)$$

Finally we estimate ϕ_5. Omitting a constant factor, we have

$$\begin{aligned}\phi_5 &= \left\{\int^n (x-u)^{h-p}\, S^p(u)\, du\right\} D_n^{h+1}\, a_{p-1/2}\,(xt) \\ &= O(n^{h+1-a}\, t^{h+1}(nt)^{-p}) \\ &= O(t^{r(p+a-h-1)\,+h+1-p}) \\ &= o(1), \qquad (3.13)\end{aligned}$$

as $t \to +0$, as in (3.9). Hence from (3.6)-(3.8), (3.12) and (3.13),

$$\phi(t) = o\,(1),$$

as $t \to +0$. This completes the proof of Theorem I.

REMARKS. (i) If p is an integer, the proof of Theorem I is much simpler. The estimate of ϕ_2 is unaffected, while, in estimating ϕ_1, we have to integrate by parts only p times, and determine the order of ϕ_6 directly.

(ii) Theorem I continues to hold for $\delta = 0$ so long as $p > 1/a$.

(iii) If in the statement of Theorem I we replace (3.2) by the corresponding o-condition, and put $p = 1$, we can deduce the following theorem of L. Schmetterer [8]: *If* $s_k = \sum_{\nu=1}^{k} a_\nu$, $S_n = \sum_{\nu=1}^{n} (s_\nu - s)$, *then* $S_n = o\,(n^{1-a})$ *and* $a_n = O(n^{-\delta})$, $1 - a \leqslant \delta < 1$, *together imply that* Σa_n *is L-summable to* s.

We next state a theorem related to Theorem I.

THEOREM II. *If* $0 < a < \delta < 1$, $p > a$ *and*

$$a_n = O(n^{-\delta}),\ n^{-p}\ S^p(n) - s = o(n^{-a}),$$

then $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* J_μ *to* s *for* $\mu = p + 1/2 - a/(1 + a - \delta)$.

As the proof is exactly similar to that of Theorem I except for details, we choose not to reproduce it here. The case $p = 0$ has been considered by K. Chandrasekharan and Otto Szász [6, Theorem 1].

4. Application to Fourier series. Let $f(x) \in L$, and be periodic with period 2π, with the Fourier expansion

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\equiv \Sigma A_n(x), \qquad (\equiv \Sigma A_n, \text{ say, for a fixed } x).$$

Let

$$g(t) \equiv g_0(t) = \tfrac{1}{2}\{f(x + t) + f(x - t) - 2s\},$$

$$g_p(t) = \frac{p}{t^p} \int_0^t (t - u)^{p-1}\, g(u)\, du,\ p > 0.$$

Let σ_n^p be the nth Cesàro mean of order p for ΣA_n. A well-known theorem on Fourier series [2] states that if $\sigma_n^p - s \to 0$ as $n \to \infty$, $p \geqslant 0$, then $g_{1+p+\epsilon}(t) \to 0$ as $t \to 0$, $\epsilon > 0$, and that this is a "best possible" result. If in the above theorem we want to strengthen the conclusion to $g_p(t) \to 0$, the hypothesis has also to be strengthened. Loo has proved [7, Theorem 2] that if

$$\sigma_n^p - s = O(n^{-a}), 0 < a < 1,$$

and $$A_n = O(n^{-\delta}), 1 - a < \delta,$$

then $g_p(t) \to 0$ as $t \to 0$, $p = 1, 2, 3, \ldots$. We prove here that Loo's theorem can be extended so as to cover *non-integral* values of p. For this we need the following

LEMMA 3. *Summability J_μ of ΣA_n to s implies that for $\mu > 1/2$,*

$$\lim_{t \to 0} g_{\mu+1/2}(t) = s.$$

For the proof see [6]. It should be understood here that $\lim_{t\to 0} g_p(t)$ has to be suitably interpreted in case $0 < p < 1 -$ either as $\lim_{t\to 0}$ ess $g_p(t)$ as in [4], or as in [1],

$$g_p(t) = \lim_{\epsilon \to 0} \frac{p}{t^p} \int_0^{t-\epsilon} (t-u)^{p-1} g(u)\, du.$$

Combining Lemma 3 with Theorem I, we deduce

THEOREM III. *If $A_n = O(n^{-\delta})$, $0 < \delta < 1$,*

and $$\sigma_n^p - s = O(n^{-a}), p > a,$$

where $0 < 1 - a < (p + \delta - 1)/p$, then $\lim_{t\to 0} g_p(t) = s$.

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ON TWO-DIMENSIONAL SUPERPOSABLE FLOWS

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INTRODUCTION. The present paper aims at discussing two-dimensional fluid motions superposable on one another. The case where one flow is rotational and the other irrotational is capable of being solved easily; and the problem of self-superposability which has been the subject of interesting studies on the continent and elsewhere in a different garb, effects considerable simplification in Navier-Stokes equations.

A general solution has been obtained for the stream function of a rotational flow, superposable on an irrotational flow, and it is found that its value depends only on the velocity potential and the stream function of the irrotational flow and the kinematic viscosity of the liquid. In the case of self-superposable flows it is seen that the vorticity is constant along the stream lines.

The fluid is supposed to be homogeneous and incompressible.

1. General derivation. The equations of motion of a viscous homogeneous incompressible fluid can be written as [1]

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - v\zeta &= -\frac{\partial \chi'}{\partial x} - \nu\frac{\partial \zeta}{\partial y}, \\ \frac{\partial v}{\partial t} + u\zeta &= -\frac{\partial \chi'}{\partial x} + \nu\frac{\partial \zeta}{\partial x}, \end{aligned}\right\} \qquad (1.1)$$

where $\chi' = p/\rho + \frac{1}{2}q^2 + \Omega$, and the symbols have their usual meanings.

To these we add the continuity condition

$$u_x + v_y = 0. \qquad (1.2)$$

Defining superposability as earlier [3], viz. that it should be possible to combine two fluid motions such that the velocity vector for the resulting fluid motion is the sum of the velocity vectors for separate motions under the same external forces, we arrive after some calculation at the equation

$$\frac{\partial}{\partial x}(u_1 \zeta_2 + u_2 \zeta_1) + \frac{\partial}{\partial y}(v_1 \zeta_2 + v_2 \zeta_1) = 0, \tag{1.3}$$

which is the condition of superposability of two fluid motions (u_1, v_1), (u_2, v_2); ζ_1, ζ_2 being the respective vortices.

Using the continuity conditions of (u_1, v_1), (u_2, v_2), (1.3) simplifies to

$$u_1 \frac{\partial \zeta_2}{\partial x} + v_1 \frac{\partial \zeta_2}{\partial y} + u_2 \frac{\partial \zeta_1}{\partial x} + v_2 \frac{\partial \zeta_1}{\partial y} = 0. \tag{1.4}$$

If ψ_1, ψ_2 be the respective stream functions, we have

$$u_r = -\frac{\partial \psi_r}{\partial y}$$

$$v_r = \frac{\partial \psi_r}{\partial x},$$

$r = 1, 2, \dots$.

The condition expressed by (1.4) can therefore be written as

$$-\frac{\partial \psi_1}{\partial y}\frac{\partial \zeta_2}{\partial x} + \frac{\partial \psi_1}{\partial x}\frac{\partial \zeta_2}{\partial y} - \frac{\partial \psi_2}{\partial y}\frac{\partial \zeta_1}{\partial x} + \frac{\partial \psi_2}{\partial x}\frac{\partial \zeta_1}{\partial y} = 0,$$

or

$$\frac{\partial(\psi_1, \zeta_2)}{\partial(x, y)} + \frac{\partial(\psi_2, \zeta_1)}{\partial(x, y)} = 0. \tag{1.5}$$

Two fluid motions (ψ_1, ζ_1), (ψ_2, ζ_2) *are therefore superposable if and only if the sum of the Jacobians of* (ψ_1, ζ_2) *and* (ψ_2, ζ_1) *vanishes.* In particular, two flows are superposable if

(i) they are irrotational, (this agrees with Rankine's idea of combining stream functions) or,

(ii) each has constant vorticity, or,

(iii) the vorticity of each is constant along the stream lines of the other.

2. One flow irrotational. If one motion, say ψ_2, is irrotational, equation (1.5) reduces to

$$\frac{\partial(\psi_2, \zeta_1)}{\partial(x, y)} = 0,$$

from which $\zeta_1 = f(\psi_2)$.

A rotational motion is therefore superposable on an irrotational motion if and only if the vorticity of the former is constant along the stream lines of the latter.

Hereafter we shall drop the suffix 2 in the case of the irrotational motion and write ζ, $\bar{\psi}$ for the vorticity and stream function of the rotational motion.

The condition of integrability of the equations (1.1) for the rotational flow is, by cross-differentiation,

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = \nu \nabla^2 \zeta, \tag{2.1}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Putting $\zeta = f(\psi)$, and remembering that ψ is irrotational, equation (2.1) becomes

$$f' \, (\psi_t + u \, \psi_x + v \, \psi_y) = \nu f'' \, [\psi_x^2 + \psi_y^2], \tag{2.2}$$

where the dashes denote differentiation with respect to ψ.

Introducing the stream function $\bar{\psi}$ of the rotational flow, equation (2.2) can be written as

$$\bar{\psi}_x \, \psi_y - \bar{\psi}_y \, \psi_x = \nu \, F(\psi) \, [\psi_x^2 + \psi_y^2] - \psi_t, \tag{2.3}$$

$$F = f''/f'.$$

Equation (2.3) determines $\bar{\psi}$.

If, in particular, $\psi_t = 0$, i.e. if the irrotational flow is steady, the auxiliary equations to solve (2.3) are

$$\frac{dx}{\psi_y} = \frac{dy}{-\psi_x} = \frac{d\bar{\psi}}{\nu \, F(\psi) \, [\psi_x^2 + \psi_y^2]}.$$

One solution of these is obviously $\psi = C$, where C is a constant. The other is obtainable from

$$\nu F(c) \{ \psi_y \, dx - \psi_x \, dy \} - d\bar{\psi} = 0. \tag{2.4}$$

Introducing the velocity potential ϕ of the irrotational motion, and remembering that the partial derivatives of ϕ and ψ satisfy Cauchy-Riemann equations, we can write (2.4) as

$$\nu F(c) \, d\phi - d\bar{\psi} = 0,$$

i.e.
$$\bar{\psi} - \nu F(\psi) \, \phi = \text{a constant}.$$

The general solution of (2.3) *when* ψ *is independent of* t *is, therefore, given by*

$$\bar{\psi} = \nu F(\psi) \, \phi + G(\psi). \tag{2.5}$$

This value of $\bar{\psi}$ has yet to fulfil the requirement

$$\zeta (= \nabla^2 \bar{\psi}) = f(\psi).$$

Now, from (2.5),

$$\nabla^2 \bar{\psi} = (\psi_x^2 + \psi_y^2) \{ G''(\psi) + \nu\phi \, F''(\psi) \}.$$

The functions G and F are, therefore, related by the equation

$$(\psi_x^2 + \psi_y^2) \{ G''(\psi) + \nu\phi \, F''(\psi) \} = f.$$

The condition that a rotational flow be superposable on a steady irrotational flow can thus be stated as follows :

The stream function ψ *of the rotational flow should be related to the stream-function* ψ *and the velocity potential* ϕ *of the irrotational flow by the equation*

$$\bar{\psi} = \nu\phi \, F(\psi) + G(\psi), \tag{2.6}$$

where $G'' = f/q^2 - \nu\phi F''$, $q^2 = \psi_x^2 + \psi_y^2$, $F = f''/f'$.

The case of a non-viscous liquid ($\nu = 0$) has already been discussed by me [5]. If $F'' = 0$, i.e. $F = (A\psi + B)$, A and B being constants, we have

$$f = c \int^{\psi} e^{(A\psi + B)^2/2A} \, d\psi,$$

c being a constant, and $q^2 = \psi_x{}^2 + \psi_y{}^2 = \dfrac{f}{G''} = h(\psi)$, say. As earlier [5], $h(\psi) = b^2\, e^{2a\psi}$, where a and b are constants, so that

$$G = \frac{c}{b^2} \int d\psi \int \left\{ e^{-2a\psi} \int^{\psi} e^{(A\psi\, +\, B)^2/2A}\, d\psi \right\} d\psi + k_1\, \psi + k_2,$$

k_1, k_2 being constants.

The value of ψ as obtained in the paper referred to above is either (i) $\alpha x + \beta y + \nu$, α, β, ν being constants, or

(ii) $- a^{-1} \log \sqrt{(a\, b\, x + D)^2 + (a\, b\, y + E)^2}$, D, E being constants. The value of ϕ is, therefore, either

(i) $\beta x - \alpha y$, or

(ii) $(1/a) \tan^{-1} \{ (a\, b\, y + E)/(a\, b\, x + D) \}$,

the additive constants being of no significance. Using (2.6), the stream function $\bar{\psi}$ of the rotational flow is given by

$$\bar{\psi} = \nu(A\psi + B)\, (\beta x - \alpha y) + \\ + \frac{C}{b^2} \int d\psi \int \left\{ e^{-2a\psi} \int^{\psi} e^{(A\psi + B)^2/2A}\, d\psi \right\} d\psi + k_1\, \psi + k_2,$$

or $$\bar{\psi} = \frac{\nu}{a}\, (A\psi + B) \tan^{-1} \left\{ \frac{a\, b\, y + E}{a\, b\, x + D} \right\} + \\ + \frac{C}{b^2} \int d\psi \left\{ e^{-2a\psi} \int^{\psi} e^{(A\psi + B)^2/2A}\, d\psi \right\} d\psi + k_1\, \psi + k_2.$$

This, therefore, furnishes an extension to viscous flows of the results obtained by me for non-viscous flows in the paper quoted above.

The constants can be determined to suit particular cases.

3. Self-superposable flows. If the fluid motion $(\bar{\psi}, \zeta)$ is self-superposable, that is, superposable on itself, equation (1.5) reduces to

$$\frac{\partial(\bar{\psi}, \zeta)}{\partial(x, y)} = 0,$$

i.e. $$\zeta = f(\bar{\psi}),$$

or $$\nabla^2 \psi = f(\bar{\psi}).$$

In the case of self-superposable flows, therefore, the vorticity is constant along the stream lines.

Further, our condition of self superposibility in a viscous or non-viscous homogeneous fluid is identical with Stokes's condition for the possibility of steady motion in a non-viscous homogeneous incompressible fluid [7].

Two dimensional self-superposable fluid flows have been studied by me in some detail earlier [4]. They have been the subject of study for others also [6] though in a different, garb—as fluid flows of which the vorticity is constant along the stream lines. The study of such flows has the advantage that it is only in their case that the non-linear terms in Navier-Stokes equations disappear.

J. Kampé de Fériet [2] also studied these flows by considering the Fourier transform

$$Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2} \int \zeta \, e^{-i(\omega_1 x + \omega_2 y)} \, d\sigma$$

of the vorticity ζ, the integral extending over the entire plane and the fluid being at rest at infinity. The general solution obtained is

$$Z(\omega_1, \omega_2, t) = Z_0(\omega_1, \omega_2) \, e^{-\nu(\omega_1^2 + \omega_2^2)t},$$

where $Z_0(\omega_1, \omega_2)$ is the arbitrary value of Z for $t = 0$. Since

$$\zeta = \int_\Omega Z(\omega_1, \omega_2, t) \, e^{i(\omega_1 \omega + x_2 y)} \, d\omega,$$

Ω being the entire plane such that

$$-\infty < \omega_1 < +\infty, \quad -\infty < \omega_2 < +\infty,$$

it appears probable that in the case of self-superposable flows the vorticity decays with time, the rate of decay depending among other factors on the kinematic viscosity of the fluid.

Glancing back at the superposable flows of category (iii) in section 1, we can express them symbolically as

$$\zeta_1 = f(\psi_2), \; \zeta_2 = g(\psi_1).$$

For the super-imposed flow the vorticity is $\zeta_1 + \zeta_2$, which is therefore equal to

$$f(\psi_2) + g(\psi_1).$$

The superimposed flow will, therefore, be self-superposable if

$$f(\psi_2) + g(\psi_1) = F(\psi_1 + \psi_2).$$

As an example, we may take, $f(\psi_2) = \psi_2$, $g(\psi_1) = \psi_1$, which fulfils the above requirement.

It may not be out of place to remark that functions satisfying the condition

$$f(A) + f(B) = f(A + B)$$

known as *additive functions,* have been studied in detail in the theory of functions of a real variable. Their properties should give a greater clue to the study of self-superposable flows.

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